technical proof

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TP B.2
Rolling resistance, spin resistance, and "ball turn"

supporting:
"The Illustrated Principles of Pool and Billiards"
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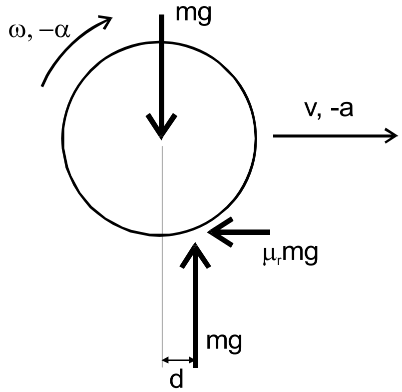
Here are the forces acting on a rolling ball:



m: ball mass $m := 6 \cdot oz$

I: ball mass moment of inertia

R: ball radius $R := 1.125 \cdot in$

μ_r: coefficient of rolling resistance

The translational equation of motion is:

$$\Sigma F_x = m \cdot a_x \qquad (1)$$

$$-\mu_r \cdot m \cdot g = m \cdot (-a)$$

The ball acceleration "a_x" is negative because the ball is decelerating. So the **coefficient of rolling resistance (CORR)** is related to the ball's deceleration according to:

$$\mu_r = \frac{a}{g} \qquad (2)$$

The rate of deceleration (a) can easily be determined experimentally by timing how long (ΔT) it takes a rolling ball to travel a certain distance (L) to a full stop.

From basic kinematics, the distance is related to deceleration and time with:

$$L = v\cdot\Delta T - \frac{1}{2}\cdot a\cdot\Delta T^2 = (a\cdot\Delta T)\cdot\Delta T - \frac{1}{2}\cdot a\cdot\Delta T^2 = \frac{1}{2}\cdot a\cdot\Delta T^2 \qquad (3)$$

so the deceleration rate can be determined with:

$$a = \frac{2\cdot L}{\Delta T^2} \qquad (4)$$

Substituting Equation 4 into Equation 2 allows us to calculate the CORR from experimental measurements:

$$\mu_r = \frac{2\cdot L}{g\cdot\Delta T^2} \qquad (5)$$

I ran a few experiments on a typical cloth and got an average value of:

$$\mu_r := 0.01$$

This number will obviously vary with cloth type and conditions. A "fast" cloth will have a lower CORR, and a slow cloth with have a higher CORR.

Returning to the diagram above, assuming the ball radius is much larger than the cloth dimple size, the rotational equation of motion (using Equation 2) is:

$$T = I\cdot\alpha$$

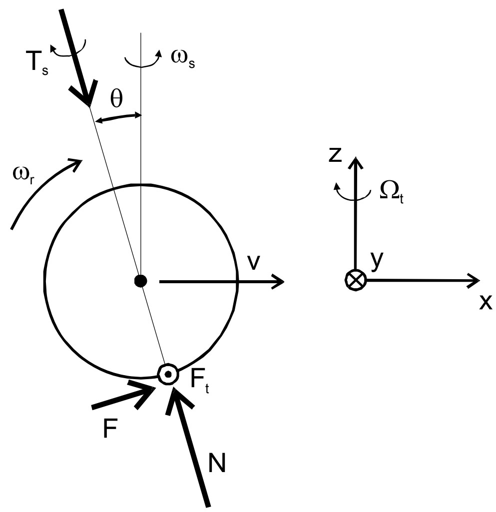
$$d\cdot m\cdot g - R\cdot\mu_r\cdot m\cdot g = \left(\frac{2}{5}\cdot m\cdot R^2\right)\cdot\left(\frac{a}{R}\right) \qquad (6)$$

$$\left(d - R\cdot\mu_r\right)\cdot mg = \frac{2}{5}\mu_r\cdot R\cdot m\cdot g$$

So, from Equation 6, the distance the vertical force is shifted forward is:

$$d := \frac{7}{5}\cdot\mu_r\cdot R \qquad d = 0.016\cdot in \qquad (7)$$

The forces in the diagram on the previous page can be represented by resultant normal force N and tangential friction force F as shown below, where the normal force acts through the center of the ball:



From Equation 6, the tangential friction force, which creates torque T, is:

$$F = \frac{T}{R} = \frac{2}{5}\mu_r \cdot m \cdot g \tag{8}$$

Equating the horizontal force components from both diagrams gives:

$$N \cdot \sin(\theta) - F \cdot \cos(\theta) = \mu_r \cdot m \cdot g \tag{9}$$

Equating the vertical components gives:

$$N \cdot \cos(\theta) + F \cdot \sin(\theta) = m \cdot g \tag{10}$$

Eliminating N from Equations 9 and 10 gives:

$$F = mg\left(\sin(\theta) - \mu_r \cdot \cos(\theta)\right) \tag{11}$$

Substituting Equation 8 gives:

$$-\mu_r \cdot \cos(\theta) + \sin(\theta) = \frac{2}{5} \cdot \mu_r \tag{12}$$

This is of the general form:

$$A \cdot \cos(\theta) + B \cdot \sin(\theta) = C \tag{13}$$

where A=-μ_r, B=1, and C=2/5μ_r. Equation 13 has the solutions (from tangent-half-angle substitutions):

$$\theta_1 = 2\,\mathrm{atan}\left(\frac{B + \sqrt{B^2 + A^2 - C^2}}{A + C}\right) \quad \text{and} \quad \theta_2 = 2\,\mathrm{atan}\left(\frac{B - \sqrt{B^2 + A^2 - C^2}}{A + C}\right) \tag{14}$$

Therefore, choosing the acute angle solution, the angle of the resultant normal is:

$$\theta = 2\text{atan}\left[\frac{1 - \sqrt{1 + \mu_r^2 - \left(\frac{2}{5}\cdot\mu_r\right)^2}}{-\mu_r + \frac{2}{5}\cdot\mu_r}\right] \qquad (15)$$

Simplifying gives:

$$\theta := 2\cdot\text{atan}\left[\frac{5}{3\cdot\mu_r}\cdot\left(\sqrt{1 + \frac{21}{25}\cdot\mu_r^2} - 1\right)\right] \qquad \theta = 0.802\cdot\text{deg} \qquad (16)$$

This is the angle at which the effective center of pressure normal force acts. I will assume that when the ball also has sidespin, the resultant forces related to rolling resistance (N and F) do not change very much.

When a ball spins in place, a friction torque develops in the cloth dimple which opposes the motion. From spin-down tests with a video camera, typical measured deceleration rates are approximately:

$$\alpha_{meas} := 10\cdot\frac{\text{rad}}{\text{sec}^2} \qquad (17)$$

This corresponds to a friction torque of:

$$T_s = I\cdot\alpha_{meas}$$

$$T_s := \frac{2}{5}\cdot m\cdot R^2\cdot\alpha_{meas} \qquad T_s = 4.917 \times 10^{-3}\cdot\text{in}\cdot\text{lbf} \qquad (18)$$

As shown in the diagram above, I will assume this spin-down torque is aligned with the center-of-pressure resultant normal vector (N) while the ball is rolling. I will also assume the torque magnitude is approximately the same as with the static spin-down test.

The shape of the ball-cloth dimple and the distribution of friction forces within the dimple are not known; but if the ball turns, the friction forces must have a sideways resultant that makes the ball turn (F_t in the diagram). If not, the analysis will predict a value of 0 for F_t. A possible explanation for this force is that the friction force on the leading edge of the dimple might be greater than the friction force on the trailing edge. The force could also represent the cloth's resistance to any tendency for masse spin to develop as a result of the torques and turning motion.

If we assume the ball's velocity turns, and if we have the z axis remain vertical and the x axis remain in-line with the ball's velocity, then the angular velocity of the ball is:

$$\vec{\omega} = \omega_r\hat{j} + \omega_s\hat{k} = \frac{v}{R}\hat{j} + \omega_s\hat{k} \qquad (19)$$

Here, I am assuming no masse spin develops while the ball is rolling. Even if there is a tendency for masse spin to develop, I am assuming cloth forces will develop (as part of F_t) to counter the spin tendency.

The angular velocity of the frame is:

$$\vec{\Omega} = -\Omega_t \hat{k} \tag{20}$$

where Ω_t is the turn rate.

If the ball's velocity turns, there must also be a turning force (F_t in the diagram) pushing the ball in the turn direction. The translational equation of motion of the ball in the y direction gives:

$$F_t = m \cdot \frac{v^2}{\rho} \tag{21}$$

where ρ is the radius of curvature of the turn. The turn rate is related to the radius of curvature with:

$$\Omega_t = \frac{v}{\rho} \tag{22}$$

Solving for ρ in Equation 22 and substituting into Equation 21 gives the following expression relating the turning force to the turn rate:

$$F_t = m \cdot v \cdot \Omega_t \tag{23}$$

The moment acting on the ball due to the reaction forces (from the original diagram, on page 1) and the spin-down friction torque (from the diagram above, on page 3) is:

$$\vec{M} = \left(T_s \sin(\theta) - F_t R \cos(\theta)\right)\hat{i} + \left(\mu_r mgR - mgd\right)\hat{j} - \left(T_s \cos(\theta) + F_t R \sin(\theta)\right)\hat{k} \tag{24}$$

The angular momentum of the ball is:

$$\vec{H} = I\vec{\omega} = \frac{2}{5} mR^2 \left(\frac{v}{R} \hat{j} + \omega_s \hat{k} \right) \tag{25}$$

The rate of change of the angular momentum, relative to the frame is:

$$\dot{\vec{H}}_{rel} = \frac{2}{5} mR^2 \left(\frac{(-a)}{R} \hat{j} + (-\alpha_s)\hat{k} \right) \tag{26}$$

where α_s is the English spin-down rate of the ball. Note that there is no x component in this equation because I assuming for now that no masse spin develops.

The rate of change of angular momentum due to rotation of the frame is:

$$\vec{\Omega} \times \vec{H} = \frac{2}{5} mRv\Omega_t \hat{i} \tag{27}$$

The equation of motion for the ball's rotation, relative to the rotating frame is:

$$\vec{M} = \frac{d}{dt}\vec{H} = \dot{\vec{H}}_{rel} + \vec{\Omega} \times \vec{H} \tag{28}$$

where the vectors are given by Equations 24, 26, and 27.

The y component of Equation 28 is the same as Equation 6, which implies the roll deceleration rate is not affected by the sidespin.

The z component of Equation 28 is similar to Equation 18, and it can be used to find the English spin-down rate for the moving ball.

The x component of Equation 28 can be used to calculate the predicted ball "turn" rate:

$$T_s \sin(\theta) - F_t R \cos(\theta) = \frac{2}{5} mRv\Omega_t \tag{29}$$

Using Equation 23, we can now solve for the ball turn rate:

$$\Omega_t = \frac{T_s \sin(\theta)}{mvR\left(\frac{2}{5} + \cos(\theta)\right)} \tag{30}$$

If masse spin were included in the analysis, Equation 30 would become:

$$\Omega_t = \frac{T_s \sin(\theta) - \frac{2}{5} mR^2 \alpha_m}{mvR\left(\frac{2}{5} + \cos(\theta)\right)} \tag{31}$$

where α_m is the rate of change of masse spin. From this equation, if masse spin develops and later degrades (i.e. α_m would change sign and magnitude), the ball could turn in either direction or go straight (i.e., Ω_t could be positive, negative, and even zero during the motion), depending upon the relative magnitude of the masse and spin-down-torque terms in the numerator. As described above (below Equation 19), we will assume no masse spin develops.

Returning to Equation 30, the ball turn rate is a function of ball speed:

$$\Omega_t(v) := \frac{1}{\left(\frac{2}{5} + \cos(\theta)\right)} \cdot \frac{T_s \cdot \sin(\theta)}{m \cdot v \cdot R} \tag{32}$$

From Equation 2, the ball's deceleration rate is:

$$a := \mu_r \cdot g \tag{33}$$

For an initial velocity v_o, the velocity of the ball as a function of time is:

$$v(v_o, t) := v_o - a \cdot t \tag{34}$$

and the time to reach a certain distance is:

$$t(v_o, x) := \frac{v_o - \sqrt{v_o^{\ 2} - 2 \cdot a \cdot x}}{a} \tag{35}$$

Therefore, the amount of **ball turn** for a shot with initial speed v_o and travel distance L is:

$$\theta_t(v_o, L) := \int_0^{t(v_o, L)} \Omega_t(v(v_o, t))\, dt \tag{36}$$

and the shot error (lateral distance from the target) would be:

$$E(v_o, L) := \int_0^{L} \theta_t(v_o, x)\, dx \tag{37}$$

Here are some example numbers for a long, slow shot (worst case) and a faster, shorter shot:

$v_o := 2 \cdot mph$ $\quad L := 8ft$ $\quad t(v_o, L) = 3.339\, s$

$\theta_t(v_o, L) = 0.305 \cdot deg$ $\quad E(v_o, L) = 0.217 \cdot in$

$v_o := 5 \cdot mph$ $\quad L := 3ft$ $\quad t(v_o, L) = 0.413\, s$

$\theta_t(v_o, L) = 0.012 \cdot deg$ $\quad E(v_o, L) = 3.811 \times 10^{-3} \cdot in$

Obviously, the "ball turn" effect is very small, as is expected from observations (e.g., NV B.7), but the physics does seem to suggest that a slow rolling ball with sidespin might tend to turn a slight amount in the direction of the spin (e.g., right spin causes turn to the right).