technical proof

TP A.4
Post-impact cue ball trajectory for any cut angle, speed, and spin

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
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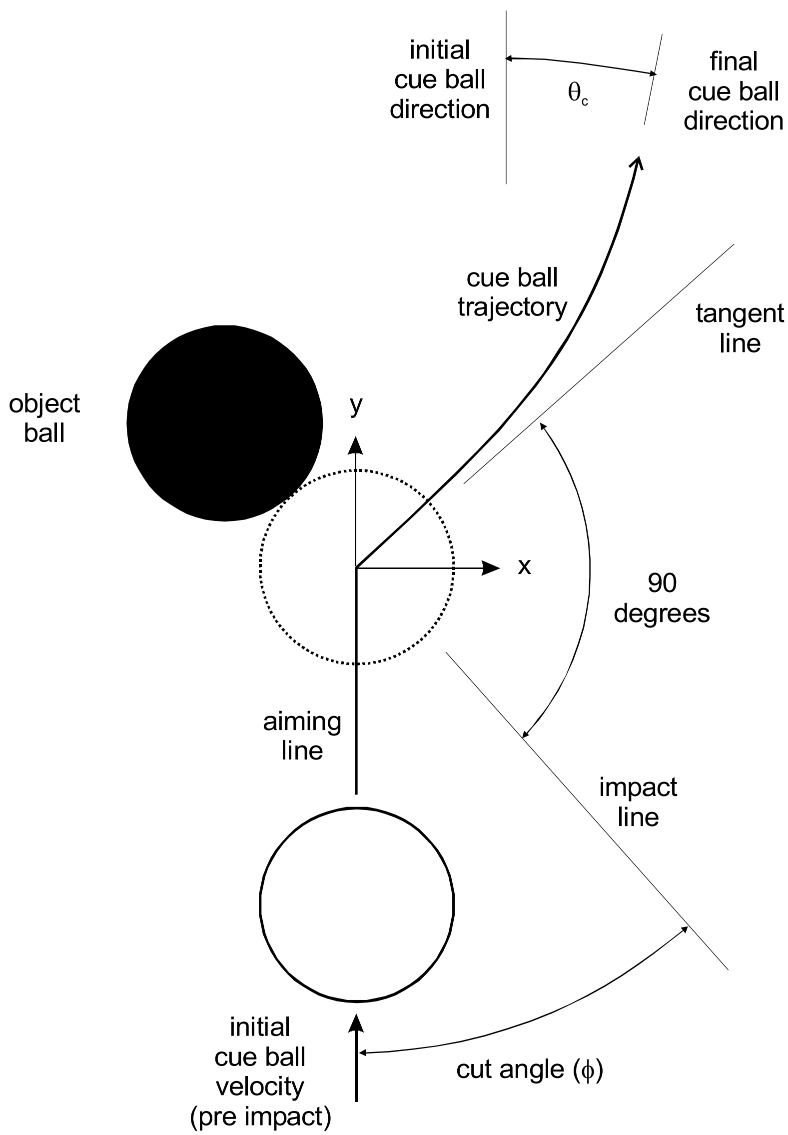


The translational equation of motion for the cue ball (after impact) is:

$$\vec{F} = m\vec{a} = m\dot{\vec{v}} \tag{1}$$

where $\vec{F}$ is the friction force between the table cloth and the cue ball during the curved trajectory and $\vec{v}$ is the velocity of the center of the cue ball.

The rotational equation of motion for the cue ball (after impact) is:

$$\vec{M} = I\vec{\alpha} = \frac{2}{5}mR^2\dot{\vec{\omega}} \tag{2}$$

where $\vec{M}$ is the moment of the friction force about the center of the cue ball

and $\vec{\omega}$ is the angular velocity of the cue ball.

The velocity of the contact point (C) between the cue ball and the table cloth is:

$$\vec{v}_C = \vec{v} + \vec{\omega} \times \vec{R} = (v_x\hat{i} + v_y\hat{j}) + (\omega_x\hat{i} + \omega_y\hat{j}) \times (-R\hat{k}) = (v_x - R\omega_y)\hat{i} + (v_y + R\omega_x)\hat{j} \tag{3}$$

Note that z-axis spin (ω_z, resulting from side English) would have no effect on the contact point velocity, and therefore would not affect the remainder of this analysis.

The friction force (μmg) opposes the slip, in a direction opposite from the relative slip velocity direction:

$$\vec{F} = -\mu mg\frac{\vec{v}_C}{|\vec{v}_C|} = -\mu mg\hat{v}_C \tag{4}$$

The friction moment can be expressed as:

$$\vec{M} = \vec{R} \times \vec{F} = -\mu mgR(\hat{R} \times \hat{v}_C) \tag{5}$$

Taking the time derivative of the left side of Equation 3, we can relate the linear and angular accelerations:

$$\dot{\vec{v}}_C = \dot{\vec{v}} + \dot{\vec{\omega}} \times \vec{R} \tag{6}$$

Substituting Equation 4 into Equation 1 gives the cue ball acceleration:

$$\dot{\vec{v}} = -\mu g\hat{v}_C \tag{7}$$

Substituting Equation 5 into Equation 2 gives the cue ball angular acceleration:

$$\dot{\vec{\omega}} = -\frac{5\mu g}{2R}(\hat{R} \times \hat{v}_C) \tag{8}$$

Therefore the last term in Equation 6 can be written as:

$$\dot{\vec{\omega}} \times \vec{R} = -\frac{5\mu g}{2R}(\hat{R} \times \hat{v}_C) \times \vec{R} = -\frac{5\mu g}{2}\hat{v}_C \tag{9}$$

Substituting Equations 7 and 9 into Equation 6 gives:

$$\dot{\vec{v}}_C = -\mu g \hat{v}_C - \frac{5\mu g}{2}\hat{v}_C = -\frac{7\mu g}{2}\hat{v}_C \tag{10}$$

The following conclusions can be made concerning Equation 10:

> The relative velocity vector, and therefore the friction force vector (see Equation 4), does not change direction!!! The relative slip speed slows to 0 and remains 0 thereafter (i.e., the cue ball starts rolling without slipping at a certain point and continues to roll in a straight line). Also, from Equations 4 and 7, the friction force vector and the cue ball acceleration are constant (in magnitude and direction). Therefore, **the cue ball trajectory will be parabolic**, just as with any constant acceleration motion (e.g., projectile motion).

From Equation 10, it is clear that the relative speed changes according to:

$$\dot{v}_C = -\frac{7\mu g}{2} \tag{11}$$

Therefore, the relative speed varies according to:

$$v_C(t) = v_{Co} - \frac{7\mu g}{2}t \tag{12}$$

where v_{Co} is the initial relative speed (immediately after object ball impact).

So, now, the relative velocity vector is known over time:

$$\vec{v}_C(t) = (v_{Co} - \frac{7\mu g}{2}t)\hat{v}_{Co} \tag{13}$$

If the initial cue ball linear and angular velocities (immediately after object ball impact) are specified, Equation 3 gives:

$$\vec{v}_C(0) = \vec{v}(0) + \vec{\omega}(0) \times \vec{R} = (v_{xo} - R\omega_{yo})\hat{i} + (v_{yo} + R\omega_{xo})\hat{j} = v_{Cox}\hat{i} + v_{Coy}\hat{j} = v_{Co}\hat{v}_{Co} \tag{14}$$

The initial relative velocity magnitude (speed) is given by:

$$v_{Co} = \sqrt{v_{Cox}{}^2 + v_{Coy}{}^2} = \sqrt{(v_{xo} - R\omega_{yo})^2 + (v_{yo} + R\omega_{xo})^2} \tag{15}$$

and the direction of the initial relative velocity (which remains constant) is:

$$\hat{v}_{Co} = \frac{(v_{Cox}\hat{i} + v_{Coy}\hat{j})}{v_{Co}} = \frac{(v_{xo} - R\omega_{yo})}{v_{Co}}\hat{i} + \frac{(v_{yo} + R\omega_{xo})}{v_{Co}}\hat{j} \tag{16}$$

Using Equation 16 in Equation 7, we now know the cue ball acceleration

$$\dot{\vec{v}} = -\mu g \hat{v}_{Co} \tag{17}$$

The solution to this equation is:

$$\vec{v}(t) = \vec{v}(0) - \mu g t \hat{v}_{Co} \tag{18}$$

This equation applies only while the cue ball is sliding. When sliding ceases, the cue ball moves in a straight line tangent to the trajectory at that point. The time it takes for sliding to cease can be found from Equation 12:

$$v_C(\Delta t) = v_{Co} - \frac{7\mu g}{2}\Delta t = 0 \tag{19}$$

So the cue ball path will be curved only for the following duration (after object ball impact)

$$\Delta t = \frac{2 v_{Co}}{7\mu g} \tag{20}$$

The final deflected angle of the cue ball path can be found by looking at the slope of the trajectory at the time given by Equation 20. From Equations 16 and 18, using Equation 20, the final components of the cue ball velocity are:

$$v_{xf} = v_x(\Delta t) = v_{xo} - \frac{\mu g (v_{xo} - R\omega_{yo})}{v_{Co}}\Delta t = v_{xo} - \frac{2}{7}(v_{xo} - R\omega_{yo}) = \frac{1}{7}\left(5v_{xo} + 2R\omega_{yo}\right) \tag{21}$$

and

$$v_{yf} = v_y(\Delta t) = v_{yo} - \frac{\mu g (v_{yo} + R\omega_{xo})}{v_{Co}}\Delta t = v_{yo} - \frac{2}{7}(v_{yo} + R\omega_{xo}) = \frac{1}{7}\left(5v_{yo} - 2R\omega_{xo}\right) \tag{22}$$

Therefore, the final deflected cue ball angle is:

$$\theta_c = \tan^{-1}\left(\frac{v_{xf}}{v_{yf}}\right) = \tan^{-1}\left(\frac{5v_{xo} + 2R\omega_{yo}}{5v_{yo} - 2R\omega_{xo}}\right) \tag{23}$$

The final ball velocity $\mathbf{v_f}$ (Equations 21 and 22) can also be expressed in the following vector form:

$$\vec{v}_f = \frac{5}{7}\vec{v}_o + \frac{2}{7}\vec{\omega}_o \times \vec{r} \tag{24}$$

where $\mathbf{v_o}$ is the initial post-impact velocity, and ω_o is the initial angular velocity. **Interestingly, the final velocity does not depend on friction μ. 5/7 (71.4%) of the final velocity comes from the initial velocity vector ($\mathbf{v_o}$) in the tangent-line direction, and 2/7 (28.6%) comes from the initial spin velocity vector (ω_o x r).** The **r** vector is straight up from the resting point to the center of the ball (i.e., **r** = **-R**, relating it to the **R** vector above).

Equation 18 can be integrated to find the x and y coordinates of the cue ball trajectory:

$$x(t) = v_{xo}t - \frac{\mu g v_{Cox}}{2v_{Co}}t^2 \tag{25}$$

$$y(t) = v_{yo}t - \frac{\mu g v_{Coy}}{2v_{Co}}t^2 \tag{26}$$

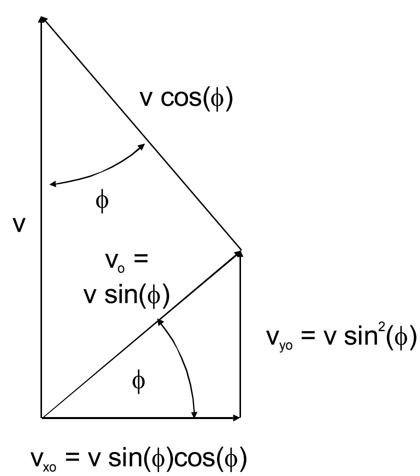
Now it is clear that the trajectory is a parabola (see also the paragraph after Equation 10). Equations 25 and 26 apply only for the time period given by Equation 20.

Given the initial cue ball speed (v) in the y direction, and neglecting the friction between the cue ball and object ball (for now), the post-impact cue ball speed and speed components are (see the figure below and TP 3.1 for more details):

$$v_o = v\sin(\phi) \tag{27}$$

$$v_{xo} = v\sin(\phi)\cos(\phi) \tag{28}$$

$$v_{yo} = v\sin^2(\phi) \tag{29}$$



If we assume the cue ball has no y-axis spin (i.e., $\omega_{yo} = 0$, which means the shot has only follow, draw, or stun), then Equation 15 (using Equations 28 and 29) becomes:

$$v_{Co} = \sqrt{v_{xo}^{\,2} + (v_{yo} + R\omega)^2} = v\sin(\phi)\sqrt{\cos^2(\phi) + \left(\sin(\phi) + \frac{R\omega}{v\sin(\phi)}\right)^2} \tag{30}$$

where ω is the initial spin of the cue ball about the x axis. And from Equation 14,

$$v_{Cox} = v_{xo} = v\sin(\phi)\cos(\phi) \tag{31}$$

$$v_{Coy} = v_{yo} + R\omega = v\sin^2(\phi) + R\omega \tag{32}$$

And from Equation 23,

$$\theta_c = \tan^{-1}\left(\frac{5v_{xo}}{5v_{yo} - 2R\omega}\right) = \tan^{-1}\left(\frac{5v\sin(\phi)\cos(\phi)}{5v\sin^2(\phi) - 2R\omega}\right) \tag{33}$$

Using Equations 28 through 32 in Equations 25 and 26, the cue ball trajectory becomes:

$$x(t) = vt\sin(\phi)\cos(\phi) - \frac{\mu g t^2 \cos(\phi)}{2\sqrt{\cos^2(\phi) + \left(\sin(\phi) + \frac{R\omega}{v\sin(\phi)}\right)^2}} \tag{34}$$

$$y(t) = vt\sin^2(\phi) - \frac{\mu g\left(\sin(\phi) + \frac{R\omega}{v\sin(\phi)}\right)t^2}{2\sqrt{\cos^2(\phi) + \left(\sin(\phi) + \frac{R\omega}{v\sin(\phi)}\right)^2}} \tag{35}$$

From Equation 20 and 30, we see that Equations 34 and 35 apply only from time 0 to time:

$$\Delta t = \frac{2v\sin(\phi)}{7\mu g}\sqrt{\cos^2(\phi) + \left(\sin(\phi) + \frac{R\omega}{v\sin(\phi)}\right)^2} \tag{36}$$

If the cue ball is rolling without slipping at object ball impact, then:

$$\omega = -\frac{v}{R} \tag{37}$$

and Equation 33 reduces to:

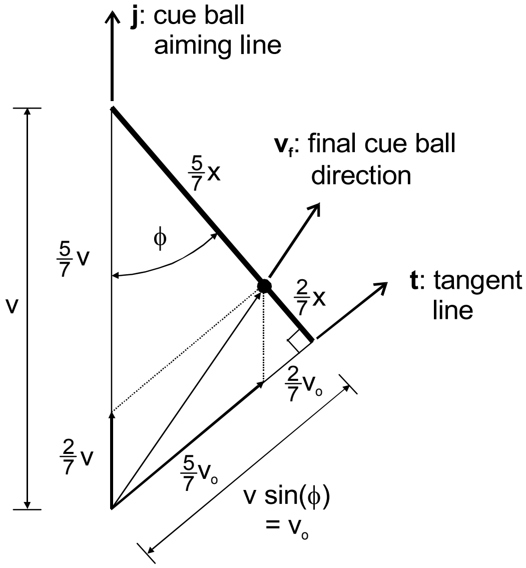
$$\theta_c = \tan^{-1}\left(\frac{\sin(\phi)\cos(\phi)}{\sin^2(\phi) + \frac{2}{5}}\right) \tag{38}$$

This agrees with the famous result from the 1987 Wallace and Schroeder paper, which formed the basis for the 30 degree rule (see TP 3.3).

Also, for a rolling cue ball, Equation 24 (using Equations 27 and 37) becomes:

$$\vec{v}_f = \frac{5}{7}\vec{v}_o + \frac{2}{7}\vec{\omega}_o \times \vec{r} = \frac{5}{7}v\sin(\phi)\hat{t} + \frac{2}{7}v\hat{j} \tag{39}$$

where **t** is the tangent line direction and **j** is the original direction of the rolling cue ball (i.e., the aiming line). The figure below shows the implications of this very useful result, using similar triangles. **The final ball direction is at 2/7 of the distance x between the tangent line and aiming line, measured perpendicular to the tangent line (i.e., parallel to the impact line).** This result is true for a rolling cue ball at any cut angle and speed.



In Bob Jewett's July '08 Billiards Digest article, he shows how you can use the cue stick to help predict the final cue ball direction for a rolling cue ball shot. If you hold the cue stick (of length "x") perpendicular to the tangent line (i.e., parallel to the impact line), with one end of the cue stick on the aiming line and the other end on the tangent line, then the final direction of the cue ball will be at the 2/7 point along the cue (at 2/7 x) from the tangent line.

Now, even though shot speed doesn't affect the final direction of the cue ball, it does affect the path to the final direction, so this also needs to be taken into consideration when predicting where the cue ball will travel (see the plots below and my June '05 Billiards Digest article).

Now we will look at ball trajectories for various types of shots. Equations 34 and 35 describe the general trajectories for follow, draw, and stun shots. They apply only during the time interval given by Equation 36. Here are the parameters used in the equations along with MathCAD form of the results:

NOTE: All parameters are expressed in metric (SI) equivalent values for dimensionless analysis

ball properties:

$$D := \frac{2.25 \cdot in}{m} \qquad R := \frac{D}{2}$$

$$D = 0.057 \qquad R = 0.029$$

coefficient of friction between the cue ball and table cloth:

$\mu := 0.2$ approximate value from several references (also backed up by my own experiment)

gravity

$$g := \frac{g}{\left(\frac{m}{s^2}\right)} \qquad g = 9.807$$

time required for cue ball to start rolling (cease sliding):

$$\Delta t(v, \omega, \phi) := \frac{2 \cdot v \cdot \sin(\phi)}{7 \cdot \mu \cdot g} \cdot \sqrt{\cos(\phi)^2 + \left(\sin(\phi) + \frac{R \cdot \omega}{v \cdot \sin(\phi)}\right)^2}$$ from Equation 36

velocity components when cue ball starts rolling in a straight line:

$$v_{xf}(v, \omega, \phi) := \frac{5}{7} \cdot v \cdot \sin(\phi) \cdot \cos(\phi)$$ from Equations 21 and 28

$$v_{yf}(v, \omega, \phi) := \frac{1}{7} \cdot \left(5 \cdot v \cdot \sin(\phi)^2 - 2 \cdot R \cdot \omega\right)$$ from Equations 22 and 29

the final deflected cue ball angle:

$$\theta_c(v, \omega, \phi) := \operatorname{atan}\left(\frac{5 \cdot v \cdot \sin(\phi) \cdot \cos(\phi)}{5 \cdot v \cdot \sin(\phi)^2 - 2 \cdot R \cdot \omega}\right)$$ from Equations 23, 28, and 29

x position of the cue ball during the curved trajectory:

$$x_c(t,v,\omega,\phi) := v\cdot t\cdot\sin(\phi)\cdot\cos(\phi) - \frac{\mu\cdot g\cdot t^2\cdot\cos(\phi)}{2\cdot\sqrt{\cos(\phi)^2 + \left(\sin(\phi) + \frac{R\cdot\omega}{v\cdot\sin(\phi)}\right)^2}}$$

from Equation 34

x position of the cue ball during and after the curved trajectory:

$$x(t,v,\omega,\phi) := \left|\begin{array}{l} \Delta T \leftarrow \Delta t(v,\omega,\phi) \\ x_c(t,v,\omega,\phi) \ \text{ if } t \le \Delta T \\ \left[x_c(\Delta T,v,\omega,\phi) + v_{xf}(v,\omega,\phi)\cdot(t-\Delta T)\right] \ \text{ otherwise} \end{array}\right.$$

y position of the cue ball during the curved trajectory:

$$y_c(t,v,\omega,\phi) := v\cdot t\cdot\sin(\phi)^2 - \left[\frac{\mu\cdot g\cdot t^2\cdot\left(\sin(\phi) + \frac{R\cdot\omega}{v\cdot\sin(\phi)}\right)}{2\cdot\sqrt{\cos(\phi)^2 + \left(\sin(\phi) + \frac{R\cdot\omega}{v\cdot\sin(\phi)}\right)^2}}\right]$$

from Equation 35

y position of the cue ball during and after the curved trajectory:

$$y(t,v,\omega,\phi) := \left|\begin{array}{l} \Delta T \leftarrow \Delta t(v,\omega,\phi) \\ y_c(t,v,\omega,\phi) \ \text{ if } t \le \Delta T \\ \left[y_c(\Delta T,v,\omega,\phi) + v_{yf}(v,\omega,\phi)\cdot(t-\Delta T)\right] \ \text{ otherwise} \end{array}\right.$$

Parameters used in plots below:

$T := 5$	number of seconds to display
$t := 0, 0.01 .. T$	0.01 second plotting increment
$\phi := 30\cdot\text{deg}$	cut angle for 1/2-ball hit
$v := 5\cdot\frac{\text{mph}}{\frac{\text{m}}{\text{s}}}$	average speed in mph converted to m/s
$\omega := -\frac{v}{R}$	roll speed
$\theta_c(v,\omega,\phi) = 33.67\cdot\text{deg}$	deflected cue ball angle

Equation for the tangent line:

$$x_{tangent\_line}(t) := \frac{t}{T}\cdot 2$$

$$y_{tangent\_line}(t) := x_{tangent\_line}(t)\cdot\tan(\phi)$$

Equation for the ball (for scale)

$$x_{ball}(t) := R\cdot\cos\left(\frac{t}{T}\cdot\frac{\pi}{2}\right)$$

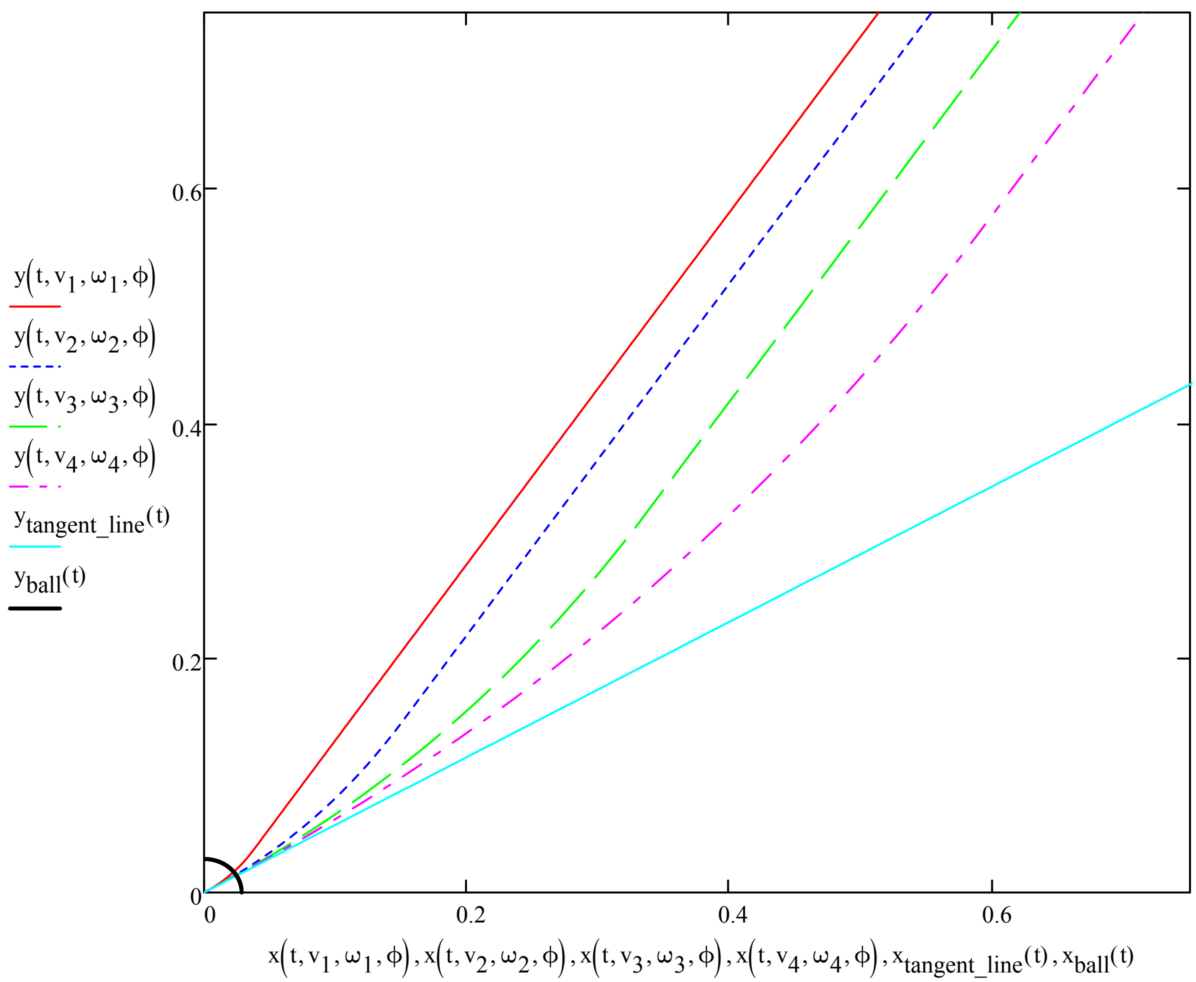
$$y_{ball}(t) := R\cdot\sin\left(\frac{t}{T}\cdot\frac{\pi}{2}\right)$$

various follow shots with natural roll

various speeds (in mph, converted to m/s) from slow to very fast:

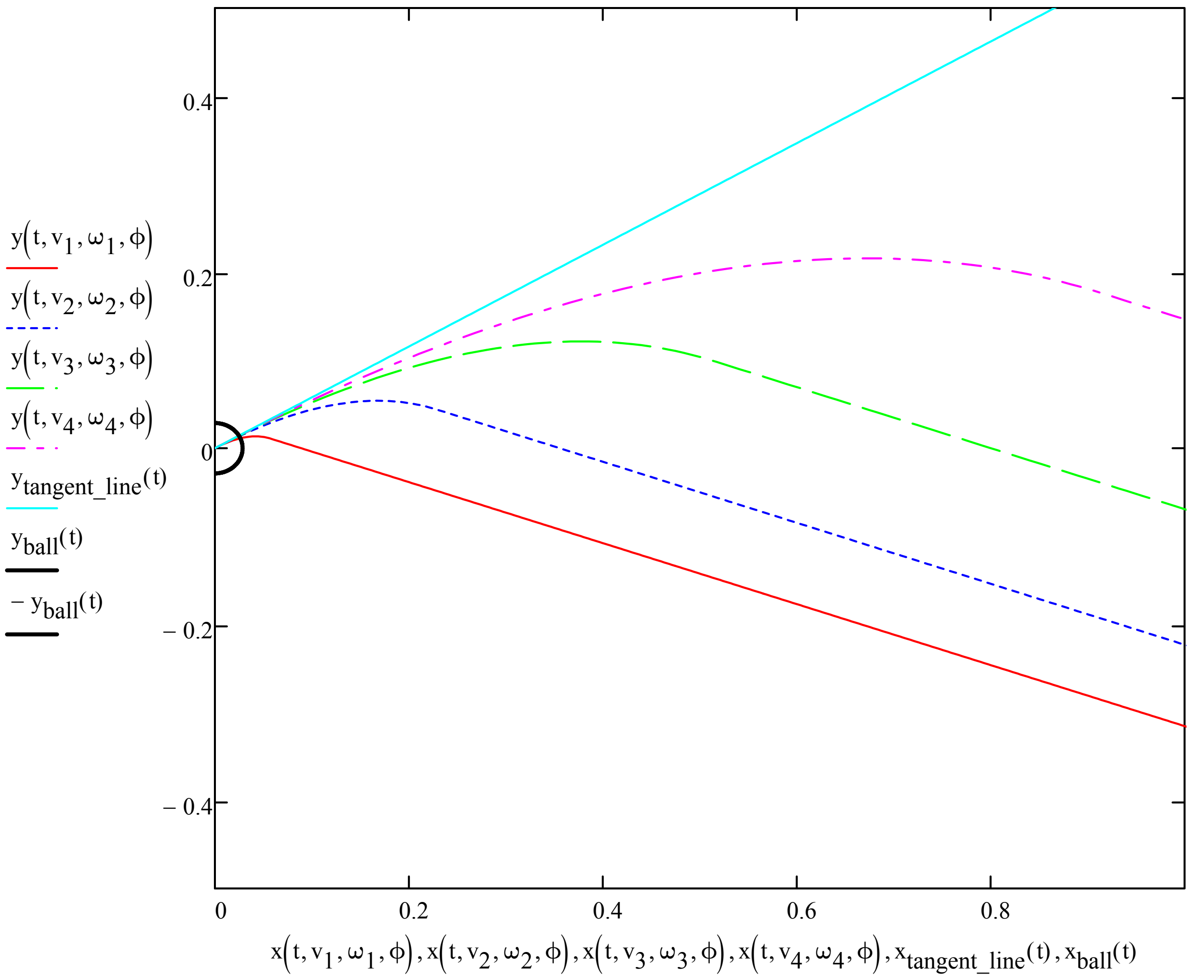
$$v_1 := 2\cdot\frac{mph}{\frac{m}{s}} \qquad v_2 := 4\cdot\frac{mph}{\frac{m}{s}} \qquad v_3 := 6\cdot\frac{mph}{\frac{m}{s}} \qquad v_4 := 8\cdot\frac{mph}{\frac{m}{s}}$$

$$\omega_1 := -\frac{v_1}{R} \qquad \omega_2 := -\frac{v_2}{R} \qquad \omega_3 := -\frac{v_3}{R} \qquad \omega_4 := -\frac{v_4}{R}$$



various draw shots with reverse natural roll

$\omega_1 := \frac{v_1}{R}$ $\omega_2 := \frac{v_2}{R}$ $\omega_3 := \frac{v_3}{R}$ $\omega_4 := \frac{v_4}{R}$



various slower speed follow shots with natural roll

$v_1 := 0.5 \cdot \dfrac{mph}{\dfrac{m}{s}}$ $v_2 := 1 \cdot \dfrac{mph}{\dfrac{m}{s}}$ $v_3 := 1.5 \cdot \dfrac{mph}{\dfrac{m}{s}}$ $v_4 := 2 \cdot \dfrac{mph}{\dfrac{m}{s}}$

$\omega_1 := -\dfrac{v_1}{R}$ $\omega_2 := -\dfrac{v_2}{R}$ $\omega_3 := -\dfrac{v_3}{R}$ $\omega_4 := -\dfrac{v_4}{R}$