technical proof

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TP A.31
The physics of squirt

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
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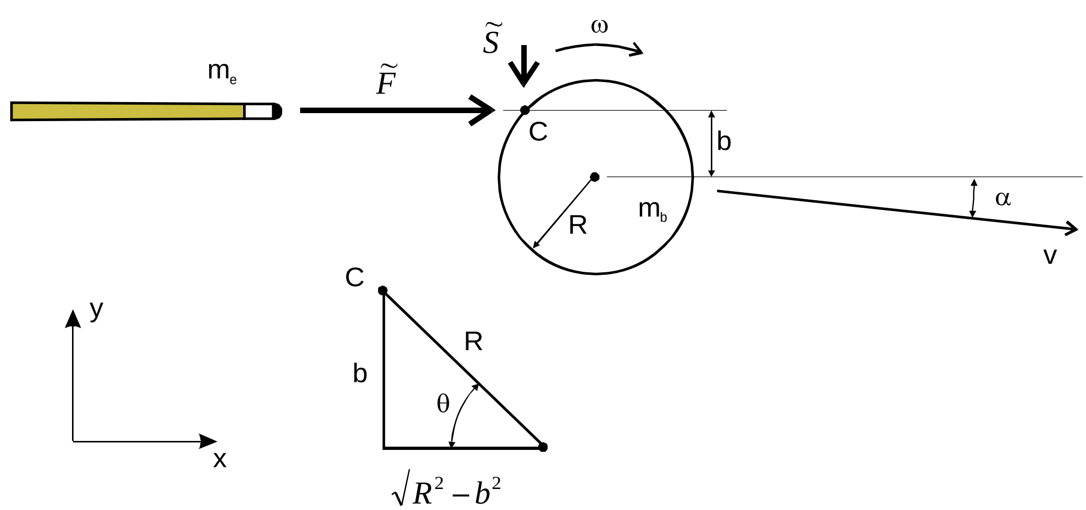
For a more detailed, and slightly different, coverage of squirt physics, see Ron Shepard's 2001 paper:
"Everything you always wanted to know about cue ball squirt, but were afraid to ask."
For more illustrations and basic explanations, see my August '07 instructional article.



$\tilde{F}$: forward impulse acting on the ball

$\tilde{S}$: side (squirt) impulse acting on the ball

v: ball speed after impact

ω: ball angular speed after impact

R: ball radius

α: squirt angle

C: point of contact

m_b : ball mass

m_e : shaft end effective mass ("endmass")

b: tip offset (Note - this is different from Shepard's "b")

Applying linear impulse and momentum principles to the ball in the x and y directions gives:

$$\tilde{F} = \int_0^{\Delta t} F(t)dt = m_b v c_\alpha \tag{1}$$

$$\tilde{S} = \int_0^{\Delta t} S(t)dt = m_b v s_\alpha \tag{2}$$

where

$$c_\alpha = \cos(\alpha) \qquad s_\alpha = \sin(\alpha) \tag{3}$$

Note that forces F and S are components resulting from both normal and friction forces acting during the brief contact time Δt. Because the tip grips the ball during contact, the friction force is difficult to model directly.

Applying linear impulse and momentum principles to the shaft in the y direction gives:

$$\tilde{S} = m_e v_{C_y} \tag{4}$$

where m_e is the effective mass of the end of the shaft (i.e., the "endmass"), and v_{Cy} is the speed of the point of contact in the y direction. Equations 2 and 4 suggest conservation of momentum in the vertical direction. The squirt momentum gained by the ball is balanced by cue stick deflection in the opposite direction. The effective mass is a function of geometry and material properties of the end of the shaft. It relates to how far the transverse elastic wave travels down the shaft (from the tip) during the brief contact time between the tip and ball. Experiments have suggested that only the last 6 inches or so of the shaft (closest to the tip) contribute to end mass. This is why cue manufactures have been successful with reducing squirt by using a smaller tip and shaft diameter, using a smaller and lighter ferrule, and drilling out the end of the shaft, all to reduce the effective endmass.

It is important to note that Equation 4 assumes the tip grips the ball. The tip is assumed to remain in contact with the ball as the ball rotates (see HSV A.76a for visual evidence of how well a chalked leather tip "grabs" the ball). While the tip and ball are in contact, the velocity of the tip and ball are equal at the point of contact. The velocity of contact point C, at the end of the impact period, can be written as:

$$\vec{v}_C = \vec{v} + \vec{\omega} \times \vec{R} = v\left(c_\alpha \hat{i} - s_\alpha \hat{j}\right) + (-\omega)\hat{k} \times \left[R\left(-c_\theta \hat{i} + s_\theta \hat{j}\right)\right] \tag{5}$$

where, from the triangle in the figure above,

$$c_\theta = \cos(\theta) = \sqrt{1 - \left(\frac{b}{R}\right)^2} \qquad s_\theta = \sin(\theta) = \frac{b}{R} \tag{6}$$

so, from Equations 5 and 6, the vertical (y or j) component of the point-of-contact velocity is:

$$v_{C_y} = R\omega c_\theta - v s_\alpha \tag{7}$$

Applying angular impulse and momentum principles to the ball gives:

$$b\tilde{F} - Rc_\theta \tilde{S} = I\omega \tag{8}$$

where I is the mass moment of inertia of the ball (2/5 $m_b R^2$).

Substituting Equation 7 into Equation 4 and equating this to Equation 2 gives:

$$m_b v s_\alpha = m_e\left(R\omega c_\theta - v s_\alpha\right) \tag{9}$$

Solving this equation for the cue ball angular speed gives:

$$\omega = \frac{v s_\alpha\left(m_b + m_e\right)}{m_e R c_\theta} \tag{10}$$

Substituting Equations 1, 2, and 10 into Equation 8 gives:

$$b m_b v c_\alpha - Rc_\theta m_b v s_\alpha = \frac{2}{5} m_b R^2\left[\frac{v s_\alpha\left(m_b + m_e\right)}{m_e R c_\theta}\right] \tag{11}$$

This equation can be rewritten as:

$$\frac{s_\alpha}{c_\alpha} = \tan(\alpha) = \frac{\left(\frac{b}{R}\right)c_\theta}{c_\theta^{\ 2} + \frac{2}{5}\left(1 + \frac{m_b}{m_e}\right)} \tag{12}$$

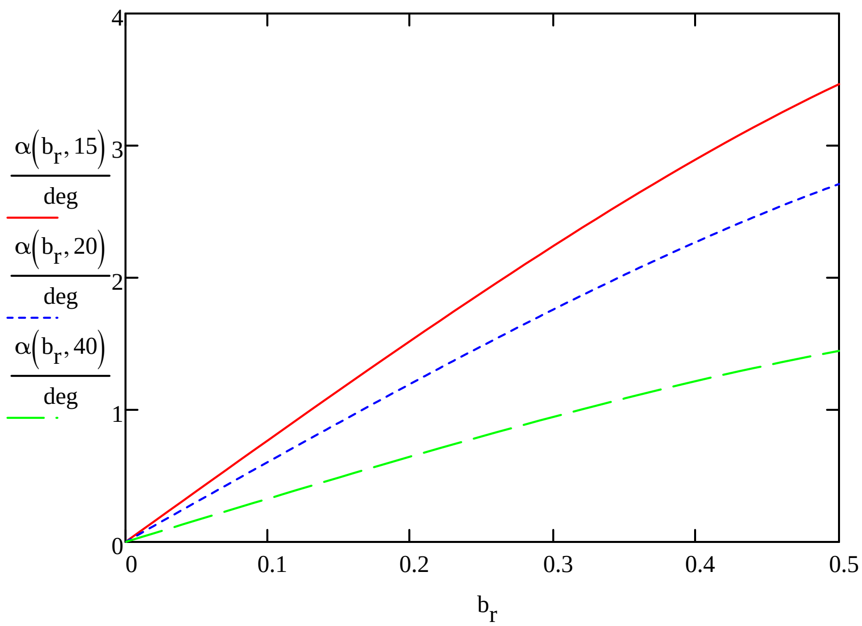
Therefore, using Equation 6, the squirt angle can be related to the tip offset and shaft endmass as:

$$\alpha = \tan^{-1}\left[\frac{\frac{5}{2}\frac{b}{R}\sqrt{1-\left(\frac{b}{R}\right)^2}}{1 + \frac{m_b}{m_e} + \frac{5}{2}\left[1-\left(\frac{b}{R}\right)^2\right]}\right] \tag{13}$$

Below is a plot of squirt angle (in degrees) vs. offset factor (b_r = b/R) for the full range of possible tip offsets (b = 0 to 0.5R) and for a typical range of ball-mass-to-endmass ratios ($m_r = m_b/m_e$), covering everything from a break cue (large endmass and small mass ratio) to a playing cue with a low-squirt shaft (small endmass and large mass ratio).

$$\alpha(b_r, m_r) := \operatorname{atan}\left[\frac{\frac{5}{2}\cdot b_r\cdot\sqrt{1-b_r^{\,2}}}{1+m_r+\frac{5}{2}\cdot\left(1-b_r^{\,2}\right)}\right]$$

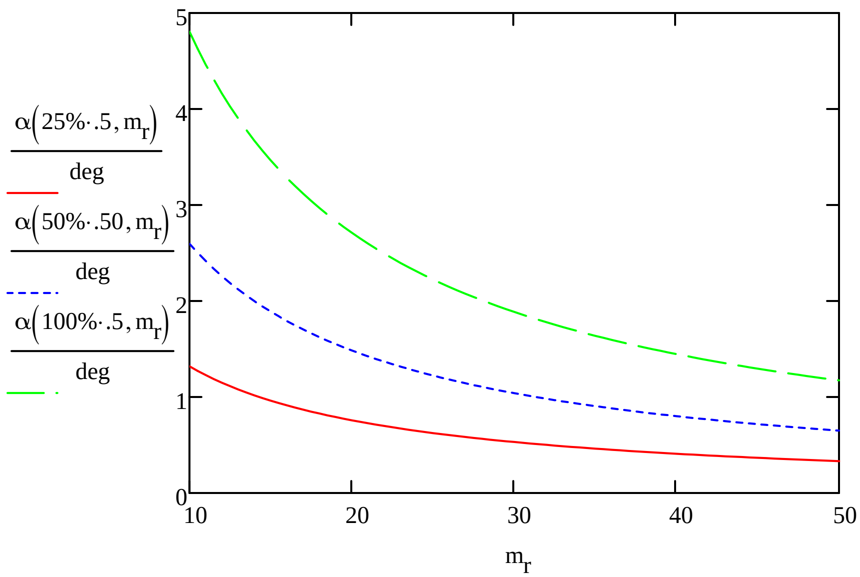
$$b_r := 0, 0.01 .. 0.5$$



The squirt angle is very close to a linear function of tip offset (i.e., the larger the offset, the larger the squirt, by a proportional amount).

Below is a plot of squirt angle (in degrees) vs. ball-mass-to-endmass ratios ($m_r = m_b/m_e$) for various amount of English (25%, 50%, 100%).

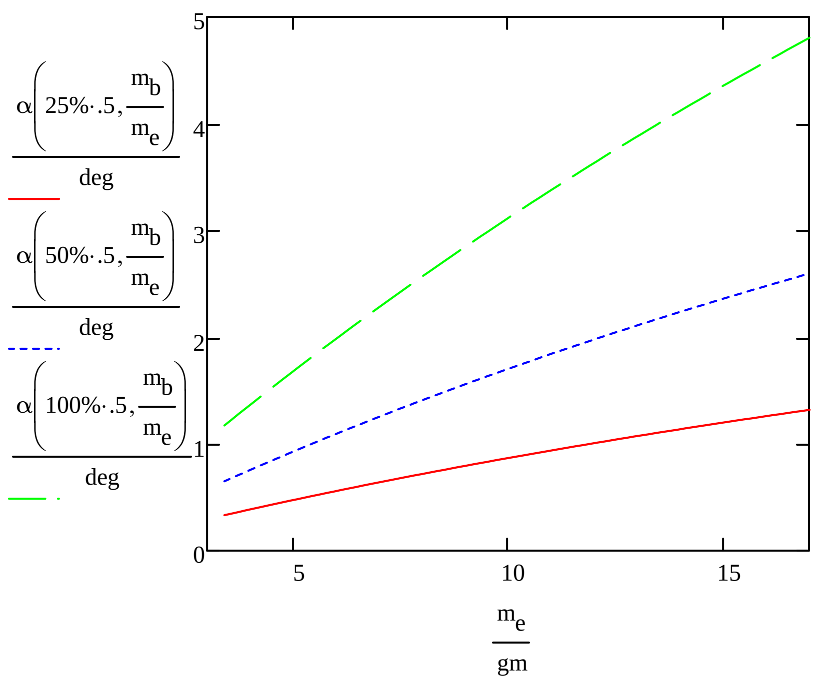
$$m_r := 10, 10.5 .. 50$$



Here is a plot of squirt (degrees) vs. endmass (grams) for different amounts of English:

$$m_b := 6 \cdot oz \qquad m_{e\_low} := \frac{m_b}{50} \qquad m_{e\_high} := \frac{m_b}{10}$$

$$m_e := m_{e\_low}, 1.05 \cdot m_{e\_low} .. m_{e\_high}$$



So it appears that squirt varies fairly linearly with effective endmass, so a certain percentage change in endmass will create close to the same percentage change in squirt.

Here is some example data for a 25% decrease in endmass:

high-squirt cue:

$$m_r := 15 \qquad \alpha(.5, m_r) = 3.466\cdot deg \qquad \alpha\left(.5, \frac{m_r}{.75}\right) = 2.709\cdot deg$$

$$\alpha(.5, m_r) - \alpha\left(.5, \frac{m_r}{.75}\right) = 0.756\cdot deg \qquad \frac{\alpha\left(.5, \frac{m_r}{.75}\right) - \alpha(.5, m_r)}{\alpha(.5, m_r)} = -21.821\cdot\%$$

low-squirt cue:

$$m_r := 40 \qquad \alpha(.5, m_r) = 1.446\cdot deg \qquad \alpha\left(.5, \frac{m_r}{.75}\right) = 1.103\cdot deg$$

$$\alpha(.5, m_r) - \alpha\left(.5, \frac{m_r}{.75}\right) = 0.343\cdot deg \qquad \frac{\alpha\left(.5, \frac{m_r}{.75}\right) - \alpha(.5, m_r)}{\alpha(.5, m_r)} = -23.714\cdot\%$$