technical proof

TP A.29
Using throw to limit cue ball motion

supporting:

"The Illustrated Principles of Pool and Billiards"

http://billiards.colostate.edu

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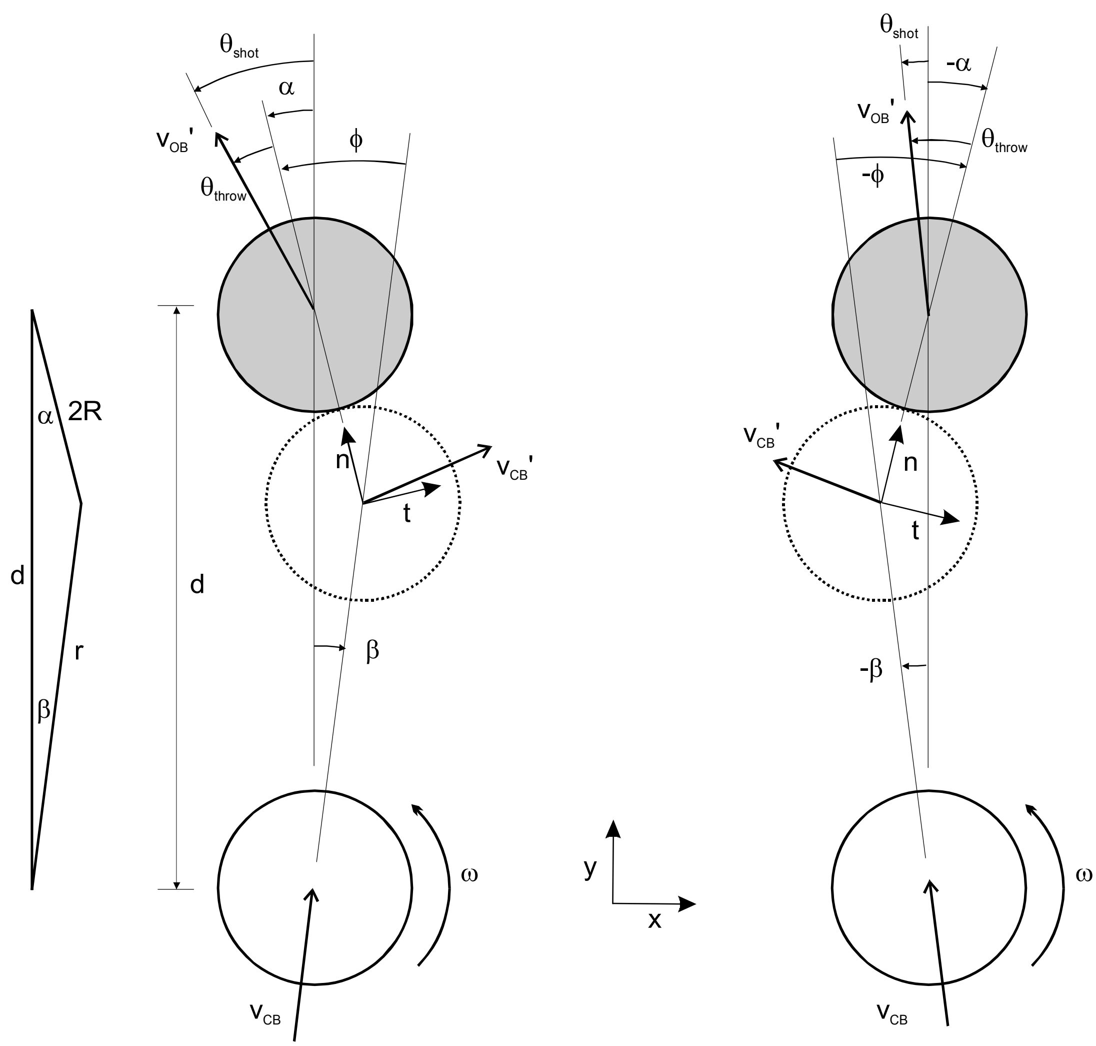
originally posted: 5/18/2007   last revision: 6/11/2015

The illustration below defines the pertinent terminology and shows how throw (θ_{throw}) is related to shot angle (θ_{shot}) and cut angle (ϕ). The cue ball has initial velocity v_{CB} and final velocity v_{CB}', and the object ball has final velocity v_{OB}'. The cue ball is assumed to have pure sidespin (ω) at contact (i.e., stun shot). A goal is to limit the cue ball motion to the right (x direction) while achieving the desired shot angle to the left.



(a) throw and cut in same directions

(b) throw and cut in opposite directions

From TP A.5, the cue ball's final speed component in the normal (n) direction is:

$$v_{CBn}' = \frac{(1-e)\cos(\phi)}{2} v_{CB} \tag{1}$$

where e is the coefficient of restitution.

Also, the change in speed in the normal direction for both balls is related to the normal impulse with:

$$\frac{\hat{F}_n}{m} = v_{OBn}' = \frac{(1+e)\cos(\phi)}{2} v_{CB} \tag{2}$$

For given ball conditions, the cut angle and amount of English create a certain amount of throw θ_{throw} (see TP A.14). The throw angle relates the tangential and normal speeds and impulses, and can be expressed (from TP A.5) as:

$$\tan(\theta_{throw}) = \frac{v_{OBt}'}{v_{OBn}'} = \frac{\hat{F}_t}{\hat{F}_n} \tag{3}$$

Therefore, using Equation 2, the change tangential speed of both balls can be expressed as:

$$\frac{\hat{F}_t}{m} = -v_{OBt}' = \frac{\hat{F}_n}{m}\tan(\theta_{throw}) = \frac{(1+e)\cos(\phi)}{2}\tan(\theta_{throw}) v_{CB} \tag{4}$$

Therefore, the cue ball's final speed component in the tangential direction (see TP A.5) is:

$$v_{CBt}' = v_{CB}\sin(\phi) + \frac{\hat{F}_t}{m} = \left[\sin(\phi) + \frac{1}{2}(1+e)\cos(\phi)\tan(\theta_{throw})\right] v_{CB} \tag{5}$$

When the cut angle (ϕ) is positive (i.e., in the same direction as the throw, as shown in figure "a" above), both the cut angle and throw terms add to the tangential velocity. When the cut angle is negative (i.e., in the opposite direction from the throw, as shown in figure "b" above), the cut-angle term subtracts from the throw portion of the tangential velocity.

For a cue-ball "hold" or "kill" shot, the goal is to have the cue ball move as little as possible to the right (in the x direction). A good measure for how much the cue ball moves to the right after contact is the x-component of the velocity after contact. I will refer to this as the cue-ball "**drift speed.**" It can be found from the n-t components, which are at angle α relative to the x-y axes:

$$v_{CBx}' = -v_{CBn}'\sin(\alpha) + v_{CBt}'\cos(\alpha) \tag{6}$$

Likewise, the object ball velocity's x-component can be found with:

$$v_{OBx}' = -v_{OBn}'\sin(\alpha) + v_{OBt}'\cos(\alpha) \tag{7}$$

For the remainder of the analysis, I will assume a perfectly elastic collision (i.e., e=1, which isn't far from the typical value of about 0.94). With this, the final cue ball drift speed can be expressed from Equations 1, 5, and 6 as:

$$v_{CBx}' = v_{CB}\cos(\alpha)\left[\sin(\phi) + \cos(\phi)\tan(\theta_{throw})\right] \tag{8}$$

To compare the cue-ball hold-effectiveness of one shot vs. another, assuming the desired shot angle (θ_{shot}) is the same, it is useful to express the cue ball's final x-direction speed as a percentage of the final object ball speed. The final object ball speed can be expressed as:

$$v_{OB}' = \sqrt{(v_{OBn}')^2 + (v_{OBt}')^2} \tag{9}$$

where the object ball's final n-t velocity components are given by Equations 2 and 4. Again, for e=1, this can be expressed as:

$$v_{OB}' = v_{CB}\cos(\phi)\sqrt{1+\tan^2(\theta_{throw})} \tag{10}$$

Therefore, the hold effectiveness can now be measured with the following drift speed, expressed as a percentage of the final object ball speed:

$$CB_{drift} = \frac{v_{CBx}'}{v_{OB}'} = \frac{\cos(\alpha)\left[\tan(\phi) + \tan(\theta_{throw})\right]}{\sqrt{1+\tan^2(\theta_{throw})}} \tag{11}$$

Similarly, using Equations 2, 4, 7, and 11, the object-ball drift percentage can be expressed as:

$$OB_{drift} = \frac{v_{OBx}'}{v_{OB}'} = \frac{-\sin(\alpha) - \cos(\alpha)\tan(\theta_{throw})}{\sqrt{1+\tan^2(\theta_{throw})}} \tag{12}$$

For a desired total shot angle θ_{shot}, and for a given amount of throw θ_{throw}, the required cut angle is (see figure "a" above):

$$\phi = \alpha + \beta \tag{13}$$

where α is:

$$\alpha = \theta_{shot} - \theta_{throw} \tag{14}$$

Angle β can be found from the triangle shown to the left of figure "a" above. The following two equations relate the vertical and horizontal distances in the triangle:

$$2R\cos(\alpha) + r\cos(\beta) = d \tag{15}$$

$$2R\sin(\alpha) = r\sin(\beta) \tag{16}$$

Solving Equations 15 and 16 for β, using Equation 14 gives:

$$\beta = \tan^{-1}\left(\frac{\sin(\theta_{shot} - \theta_{throw})}{\frac{d}{2R} - \cos(\theta_{shot} - \theta_{throw})}\right) \tag{17}$$

So, from Equations 13, 14, and 17, the cut angle is related to the shot and throw angles with:

$$\phi = \theta_{shot} - \theta_{throw} + \beta = \theta_{shot} - \theta_{throw} + \tan^{-1}\left(\frac{\sin(\theta_{shot} - \theta_{throw})}{\frac{d}{2R} - \cos(\theta_{shot} - \theta_{throw})}\right) \tag{18}$$

To see if throw can be used to help kill cue ball motion for a shot with a required angle θ_{shot}, we can compare the cue ball's drift speed with throw to the drift speed without throw (e.g., if gearing outside English is used, or if CIT is neglected). With no throw (θ_{throw}=0), using Equations 11, 14, and 18, the drift-speed percentage can be expressed as:

$$CB_{drift_{no\_throw}} = \cos(\theta_{shot})\tan(\theta_{shot} + \beta) \tag{19}$$

With throw, the drift-speed percentage can be expressed as:

$$CB_{drift_{with\_throw}} = \frac{\cos(\theta_{shot} - \theta_{throw})\left[\tan(\theta_{shot} - \theta_{throw} + \beta) + \tan(\theta_{throw})\right]}{\sqrt{1 + \tan^2(\theta_{throw})}} \tag{20}$$

The drift-speed percentage, with throw, can actually be negative, implying that the cue ball actually drift in the same direction as the shot (to the left). From the bracketed term in the numerator, this occurs when β is less than the negative of the shot angle θ_{shot} (see more below).

Here is the MathCAD implementation of Equations 17, 19, and 20:

$D := 2.25\cdot in$ $\quad R := \frac{D}{2}$ $\quad$ ball diameter and radius

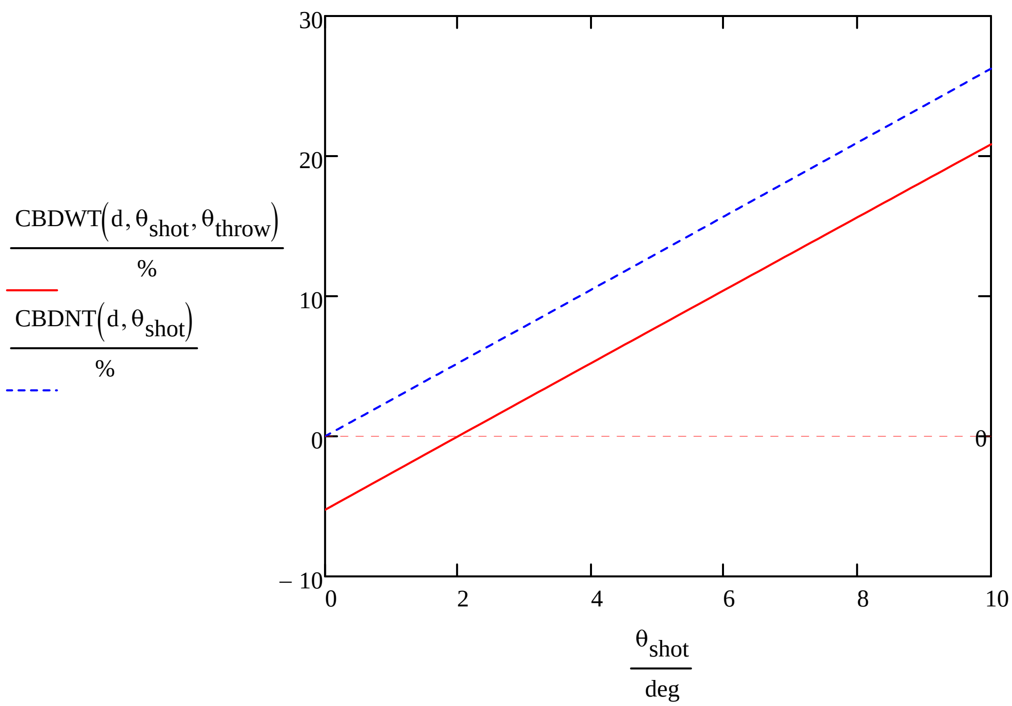
$$\beta(d, \theta_{shot}, \theta_{throw}) := \operatorname{atan}\left(\frac{\sin(\theta_{shot} - \theta_{throw})}{\frac{d}{2\cdot R} - \cos(\theta_{shot} - \theta_{throw})}\right)$$

$$CBDWT(d, \theta_{shot}, \theta_{throw}) := \frac{\cos(\theta_{shot} - \theta_{throw})\cdot\left(\tan(\theta_{shot} - \theta_{throw} + \beta(d, \theta_{shot}, \theta_{throw})) + \tan(\theta_{throw})\right)}{\sqrt{1 + (\tan(\theta_{throw}))^2}}$$

$$CBDNT(d, \theta_{shot}) := \cos(\theta_{shot})\cdot\tan(\theta_{shot} + \beta(d, \theta_{shot}, 0))$$

A typical value for the maximum amount of throw possible, with a slow-speed stun-shot with the optimal amount of English, is about 6 degrees. Using this, we can see how the drift-speed percentages vary with the shot angle. For this example, we will use a shot distance of 3 ball diameters.

$$d := 3 \cdot D \qquad \theta_{throw} := 6 \cdot deg \qquad \theta_{shot} := 0 \cdot deg, 0.1 \cdot deg .. 10 \cdot deg$$



Regardless of the shot angle, the cue ball drift will be less if throw is used, as compared to not using throw (see the two curves above). For small shot angles, if the throw angle is large enough, the cue ball can be stopped completely and even be made to drift in the same direction as the shot (i.e., the drift is negative). For the example above, the maximum shot angle possible is:

$$\theta_{shot} := 2 \cdot deg$$

Given

$$CBDWT(d, \theta_{shot}, \theta_{throw}) = 0$$

$$\theta_{shot} := Find(\theta_{shot})$$

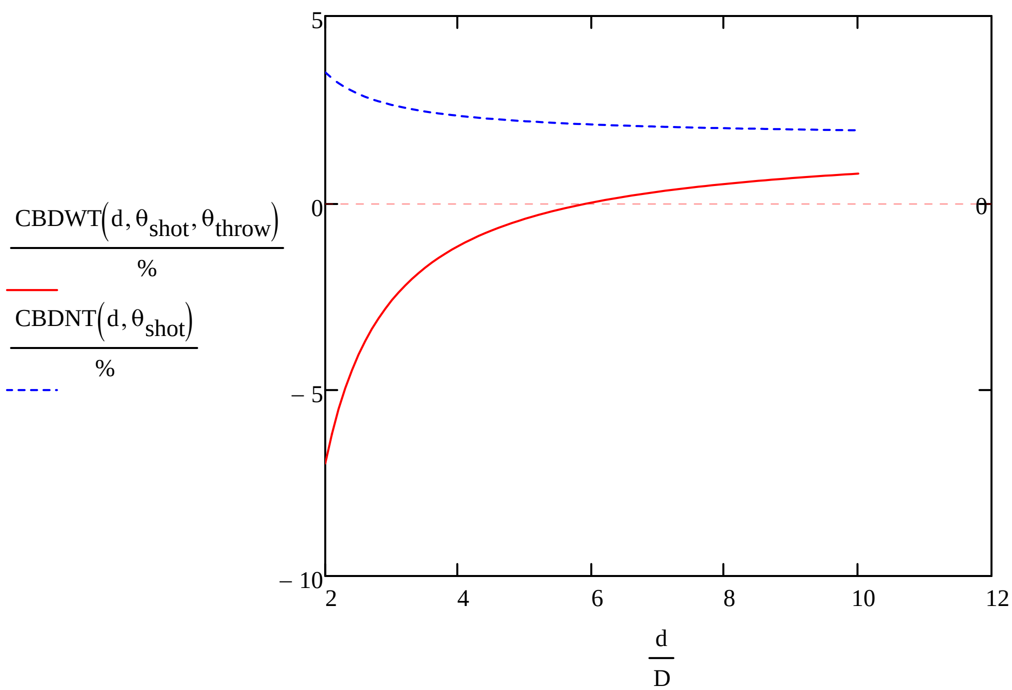
$$\theta_{shot} = 1.997\ deg$$

From Equation 11, the limit where negative cue ball drift is possible is always where the cut angle ϕ is the negative of the throw angle θ_{throw}, or from Equation 18, where β is the negative of the shot angle θ_{shot}:

$$\beta(d, \theta_{shot}, \theta_{throw}) = -1.997\ deg$$

We can also look at how the drift speed percentages vary with distance between the cue ball and object ball, for a given shot angle and throw amount:

$\theta_{shot} := 1 \cdot deg \qquad d := 2 \cdot D, 2.1 \cdot D .. 10 \cdot D$



So when the cue ball is closer to the object ball, it is much easier to use throw to reduce (and even reverse) the cue ball drift. However, even for the small shot angle (1 deg) and large throw angle (6 deg) above, it is impossible to reverse or prevent cue ball drift once the balls are past a certain distance apart. For the example above, the distance is about 6 ball diameters (see the graph above or the calculation below).

$d := 6 \cdot D \qquad \theta_{shot} := 1 \cdot deg$

Given

$$CBDWT(d, \theta_{shot}, \theta_{throw}) = 0$$

$d := Find(d)$

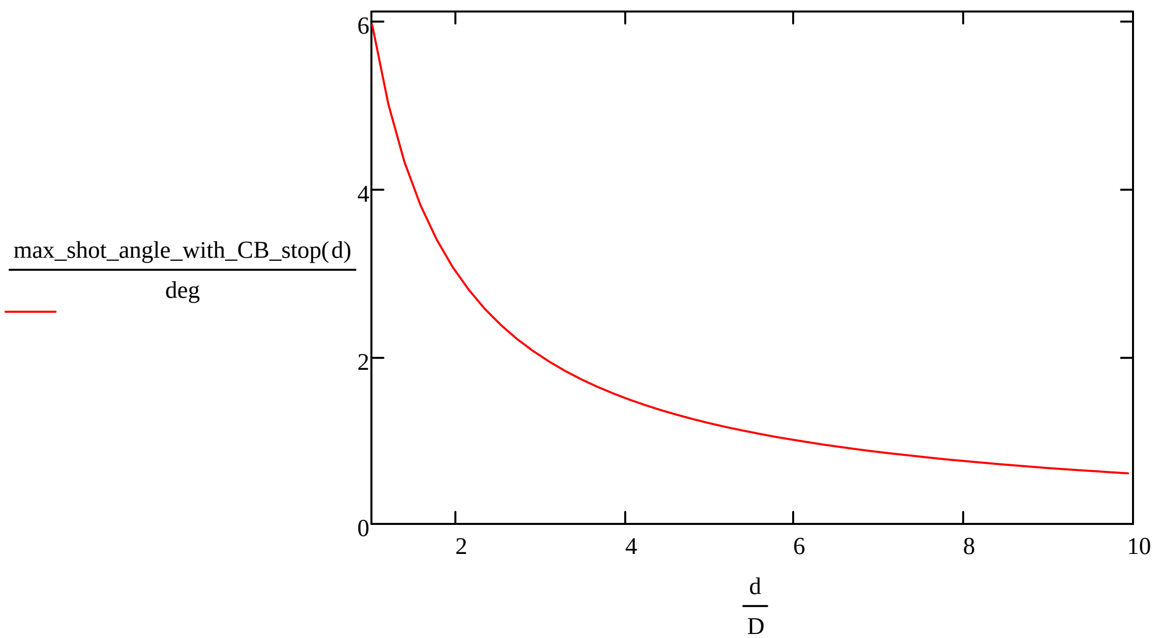
$\frac{d}{D} = 5.989 \qquad d = 13.476\,in$

An interesting challenge shot is to try to create an OB shot angle (θ_{shot}>0) while stopping the CB in place (CBDWT=0). Below is a plot of the maximum shot angle possible for different distances (d) between the CB and OB, assuming a typical maximum throw value of 6 degrees.

$\theta_{throw} := 6 \cdot deg$

$max\_shot\_angle\_with\_CB\_stop(d) := root\left(CBDWT\left(d, \theta_{shot}, \theta_{throw}\right), \theta_{shot}\right)$

$d := 1.01 \cdot D, 1.2 \cdot D .. 10 \cdot D$



The maximum shot angle possible approaches the maximum amount of throw possible as the CB gets closer to the OB (d = D). As the distance (d) between the CB and OB increases, the maximum shot angle possible becomes less.

Theoretically (neglecting swerve), per the plot above, it is possible to hold the CB and create an OB angle at any CB-OB distance. However, at the table, there is a practical limit to the CB-OB distance over which this can be done. The analysis above has ignored the effects of swerve. Swerve changes the effective cut angle of the shot, making it more difficult to hold the CB. And at larger distances, shot swerve becomes more of a factor. Also, stun (for maximum throw) is more difficult to control at larger distances. Also, it is much more difficult to judge squirt and swerve and be accurate with the hit at larger distances.

Please refer to the "throw-CB hold" resource page in the FAQ section at billiards.colostate.edu for illustrations, examples, and video demonstrations.