technical proof

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TP A.27
Spin Transfer

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
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For illustrations and background information, see TP 4.4.

This analysis is for a straight-on (0 deg cut angle) stop shot with English. The cue ball (CB) throws and transfers spin to the (OB).

From Equation 8 in TP A.5, the normal impulse (transferred linear momentum) between the CB and OB during impact is:

$$\widetilde{F}_n = \frac{(1+e)}{2} mv \tag{1}$$

where e is the coefficient of restitution and m and v are the mass and initial speed of the CB.

Friction between the CB and OB during impact creates forces in the tangential direction also. From TP 4.3, the tangential impulse is related to the normal impulse according to:

$$\widetilde{F}_t = \widetilde{F}_n \tan\theta = \frac{(1+e)}{2} mv \tan\theta \tag{2}$$

where θ is the throw angle.

The tangential impulse, in addition to creating throw, transfers spin from the CB to the OB. From angular impulse and momentum principles,

$$\widetilde{F}_t R = I\Delta\omega \tag{3}$$

where $\Delta\omega$ is the amount of spin transfer and I is the moment of inertia of the ball (2/5 mR^2). From conservation of angular momentum, the amount of spin transferred to the OB is the same as the amount of spin lost by the CB.

Therefore, from Equations 2 and 3, the amount of spin transferred to the OB is:

$$\Delta\omega = \frac{\tilde{F}_t R}{I} = \frac{(1+e)}{2}\frac{mvR\tan\theta}{\frac{2}{5}mR^2} = \frac{5(1+e)}{4}\left(\frac{v}{R}\right)\tan\theta \tag{4}$$

From TP A.14, for the limiting case of throw where there is no slipping between the CB and OB at the end of the collision (which is where the maximum spin transfer occurs),

$$\tan(\theta) = \frac{R\omega}{7v_{OB}} = \frac{1}{7}\frac{\omega}{\left(\frac{v_{OB}}{R}\right)} \tag{5}$$

where ω is the initial spin rate of the CB, and v_{OB} is the final speed of the OB.

From TP A.15 (Equation 9), the final OB speed is:

$$v_{OB} = \frac{(1+e)}{2}v \tag{6}$$

where v is the initial speed of the CB. Therefore, from Equation 5, the initial CB spin rate can be related to the throw with:

$$\omega = 7\frac{(1+e)}{2}\left(\frac{v}{R}\right)\tan(\theta) \tag{7}$$

Using Equations 4 and 7, we can now solve for the maximum possible **spin transfer percentage (STP)**:

$$STP_{\max} = \frac{\Delta\omega}{\omega} = \frac{5/4}{7/2} = \frac{5}{14} = 35.71\% \tag{8}$$

Notice how this result does not depend on the coefficient of restitution (e) of the balls! This is the limiting spin transfer percentage, regardless of the ball elasticity.

More generally (but assuming e=1), from TP A.14,

$$\mu\left(\mu_{factor}, v\right) := \mu_{factor}\cdot\left(9.951\cdot 10^{-3} + 0.108\cdot e^{-1.088\cdot v}\right)$$ ball friction

where μ_{factor} is used to account for various ball conditions (μ_{factor}=1 is for average ball friction)

$$v_{rel}\left(v, \omega_x, \omega_z, \phi\right) := \sqrt{\left(v\cdot\sin(\phi) - R\cdot\omega_z\right)^2 + \left(R\cdot\omega_x\cdot\cos(\phi)\right)^2}$$ relative speed between ball surfaces

$$\tan\theta\left(\mu_{factor}, v, \omega_x, \omega_z, \phi\right) := \frac{\min\left(\begin{array}{c}\dfrac{\mu\left(\mu_{factor}, v_{rel}\left(v, \omega_x, \omega_z, \phi\right)\right)\cdot v\cdot\cos(\phi)}{v_{rel}\left(v, \omega_x, \omega_z, \phi\right)} \\ \dfrac{1}{7}\end{array}\right)\cdot\left(v\cdot\sin(\phi) - R\cdot\omega_z\right)}{v\cdot\cos(\phi)}$$

Therefore, from Equation 4, the spin transfer percentage for any straight-on shot with English is:

$$\mathrm{STP} = \frac{\Delta\omega}{\omega}$$

$$\mathrm{STP}\left(\mu_{factor}, v, \omega_x, \omega_z, \phi\right) := \frac{1}{\omega_z}\cdot\frac{5}{2}\cdot\frac{v}{R}\cdot\left|\tan\theta\left(\mu_{factor}, v, \omega_x, \omega_z, \phi\right)\right|$$

From TP A.12 and TP A.25, English rate of spin (ω_z) is related to percent English (PE) and shot speed according to:

$$\omega_z(PE, v) := 1.25\cdot PE\cdot\frac{v}{R}$$

For a straight-on stun shot, the amount of spin-induced throw and spin transfer depends on the shot speed (see the plots on the next page):

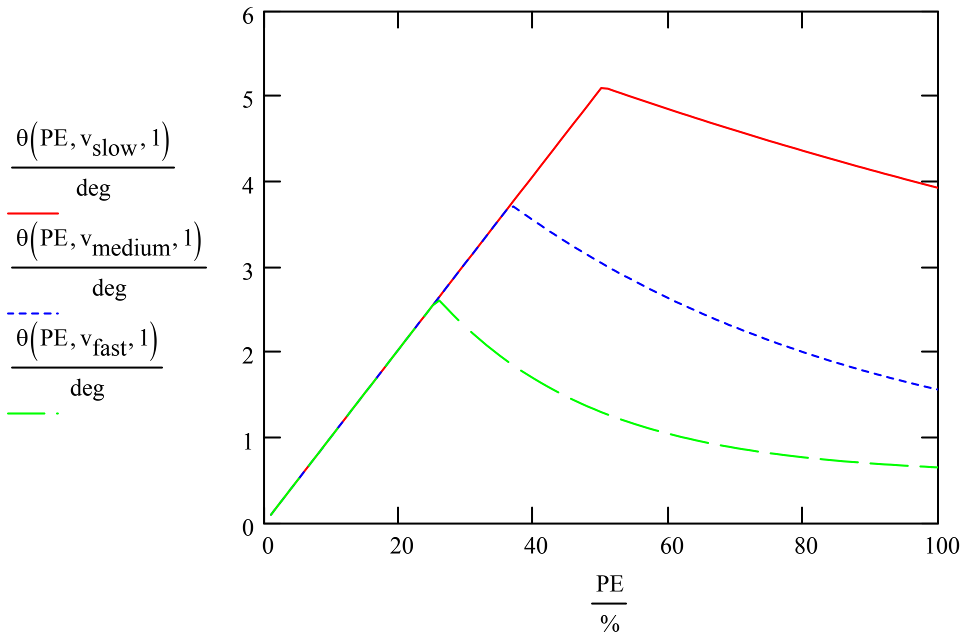
$$\phi := 0 \qquad \omega_x := 0 \qquad R := \frac{1.125\cdot in}{m}$$

$$v_{slow} := \frac{1\cdot mph}{\frac{m}{s}} \qquad v_{medium} := \frac{3\cdot mph}{\frac{m}{s}} \qquad v_{fast} := \frac{7\cdot mph}{\frac{m}{s}}$$

$$PE := 0\cdot\%, 1\% .. 100\%$$

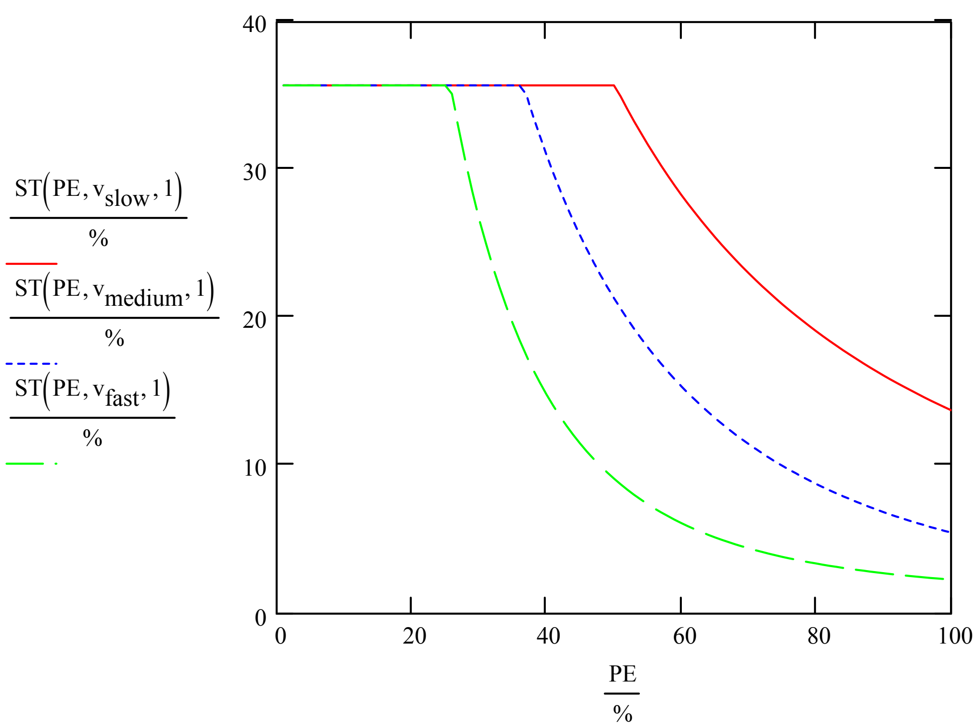
throw angle vs. percent English at different speeds:

$$\theta\left(PE, v, \mu_{factor}\right) := \operatorname{atan}\left(\tan\theta\left(\mu_{factor}, v, \omega_x, -\omega_z(PE, v), \phi\right)\right)$$



spin transfer percentage vs. percent English at different speeds:

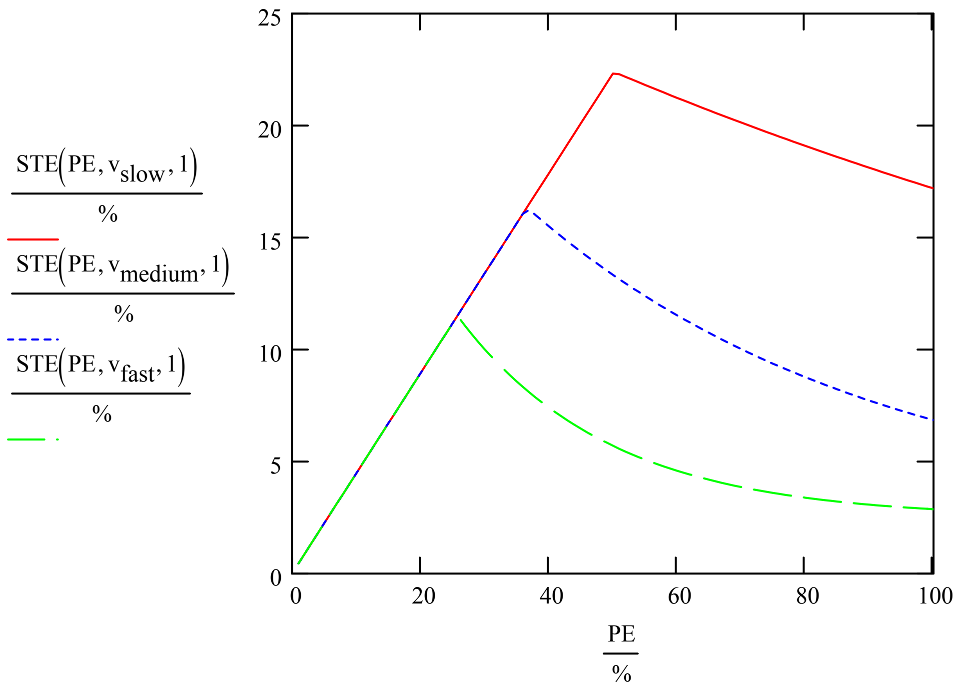
$$ST\left(PE, v, \mu_{factor}\right) := STP\left(\mu_{factor}, v, \omega_x, \omega_z(PE, v), \phi\right)$$



When looking at spin transfer, spin transfer percentage can be a misleading measure. A more useful measure is the amount of spin ($\Delta\omega$) as compared to the ball speed (v/R). This measure is what is important in determining the amount of rebound angle change off a rail cushion. From Equation 4, the **spin transfer effectiveness (STE)** can be defined as:

$$\text{STE} = \frac{\Delta\omega}{\left(\frac{v}{R}\right)}$$

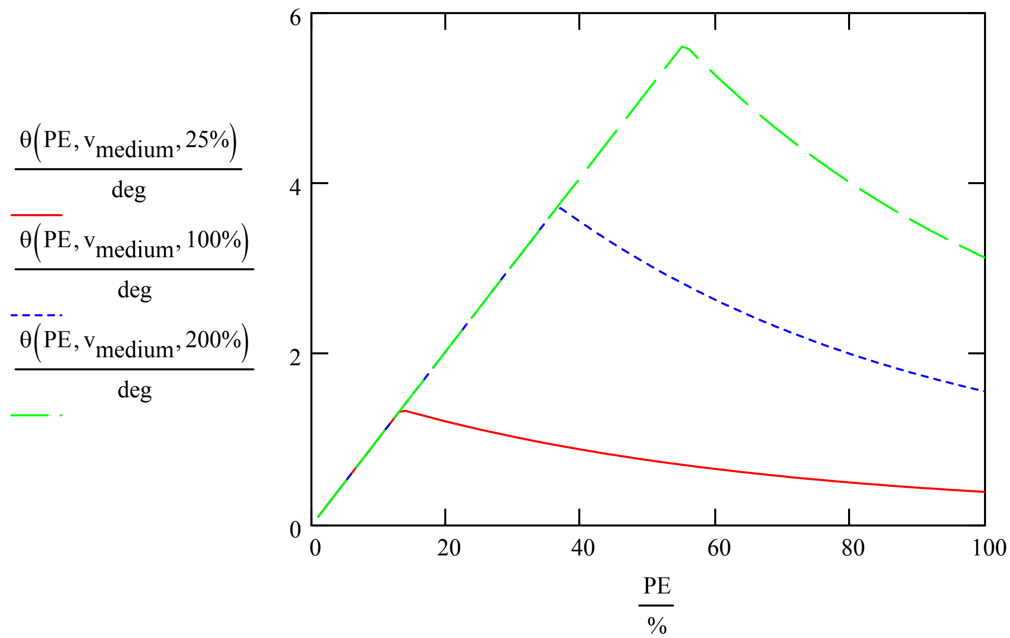
$$\text{STE}\left(\text{PE}, v, \mu_{\text{factor}}\right) := \frac{5}{2}\cdot\tan\theta\left(\mu_{\text{factor}}, v, \omega_x, -\omega_z(\text{PE}, v), \phi\right)$$



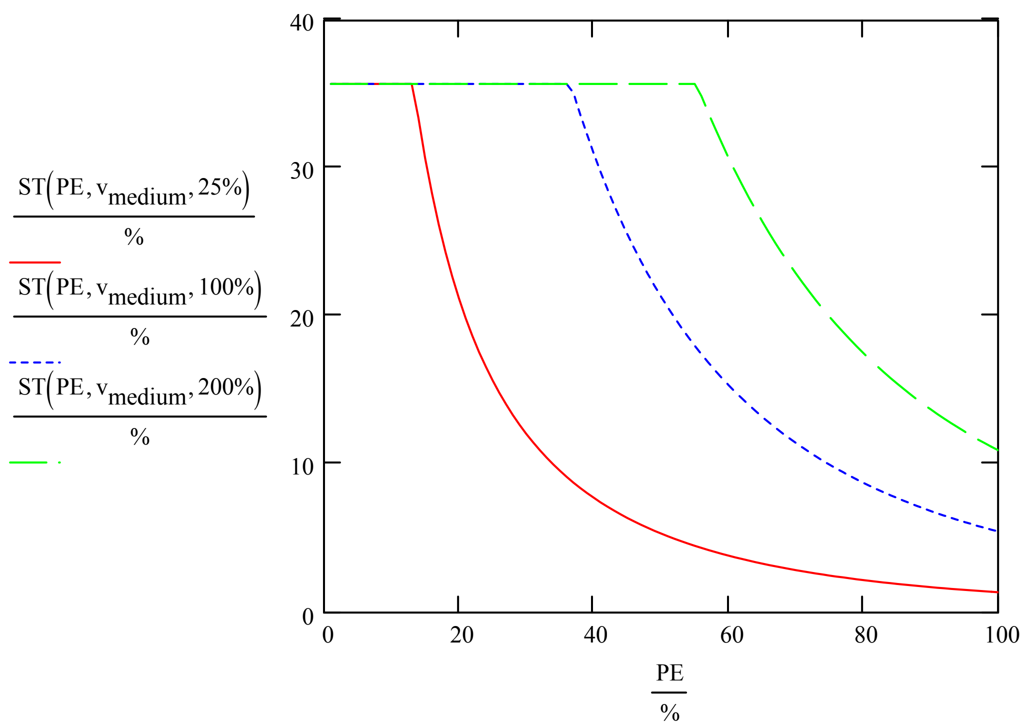
Therefore, for a slow to medium speed shot, the spin transfer is most effective in the 40-50% English range, where the spin transfer can be as high as 35.71%.

Now, let's look at what happens for different ball conditions. The plots below show how the results change for balls with a lot less friction (25%) and a lot more friction (200%), as compared to average friction (100%), for a medium speed shot.

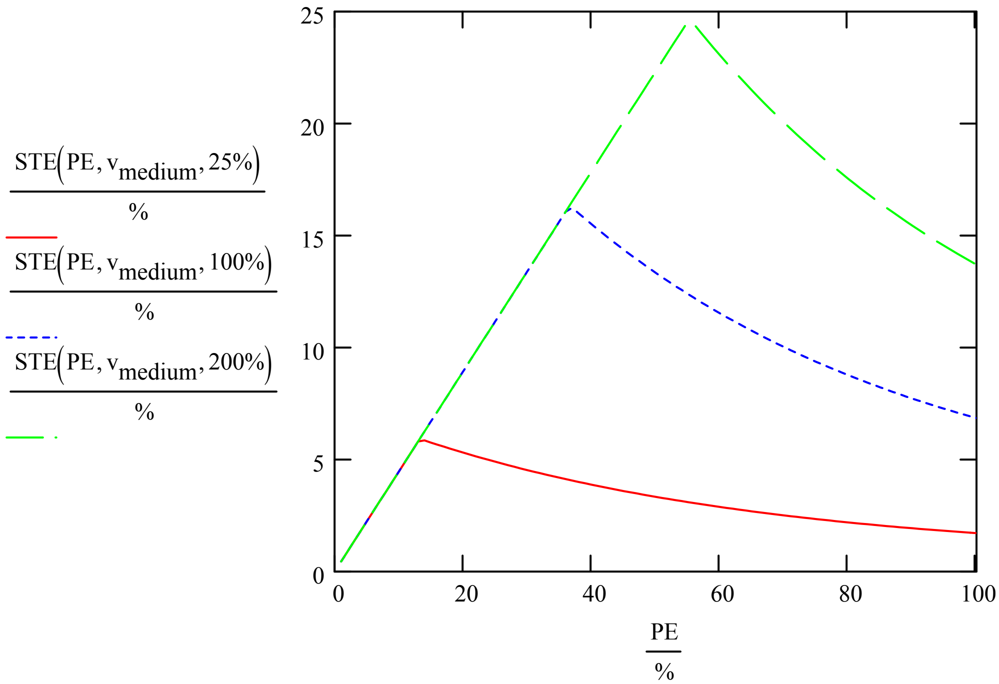
throw angle vs. percent English under different ball conditions:



spin transfer percentage vs. percent English under different ball conditions:



spin transfer effectiveness vs. percent English under different ball conditions:



Therefore, for low friction (new, clean, and/or polished) balls, the spin transfer effectiveness is very low. On the other hand, clingy balls can transfer spin much more effectively.