technical proof

TP A.16
Final ball speeds, distances, and directions for natural roll shots

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
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See TP 3.1, TP 3.2, and TP A.4 for background and illustrations.

For a stun shot, see TP 3.2.

From TP 3.2 and TP 4.1, the final object ball (OB) speed, given the initial cue ball (CB) speed v and the the cut angle ϕ is:

$$v_{OB}(v,\phi) := \frac{5}{7} v \cdot \cos(\phi)$$

Note: OB spin-transfer and swerve effects (see TP A.24) are being neglected here (because the effects are so small).

From TP A.4, the velocity components for the cue ball after post-impact rolling begins (i.e., after sliding and curving ceases) is:

$$v_{CBx}(v,\phi) := \frac{5}{7} \cdot v \cdot \sin(\phi) \cdot \cos(\phi)$$

$$v_{CBy}(v,\phi) := \frac{1}{7} \cdot \left(5 \cdot v \cdot \sin(\phi)^2 + 2 \cdot v\right)$$

Therefore, the final cue ball speed is:

$$v_{CB}(v,\phi) := \sqrt{v_{CBx}(v,\phi)^2 + v_{CBy}(v,\phi)^2}$$

From TP 3.3, the cut angle and ball-hit fraction are related by:

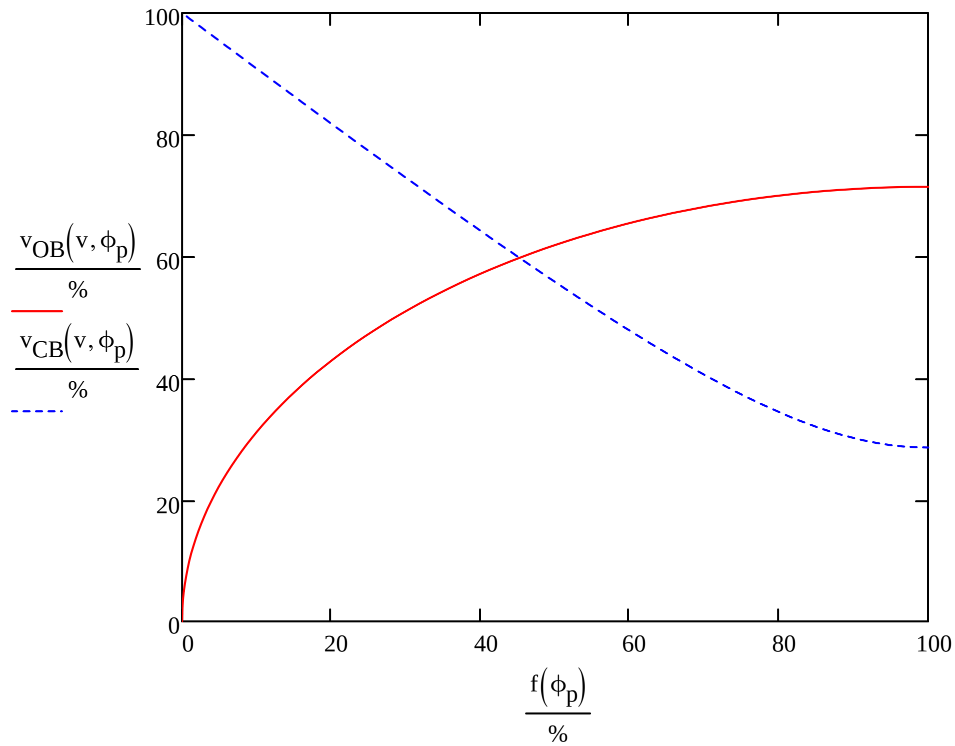
$$f(\phi) := 1 - \sin(\phi) \qquad \phi(f) := \operatorname{asin}(1 - f)$$

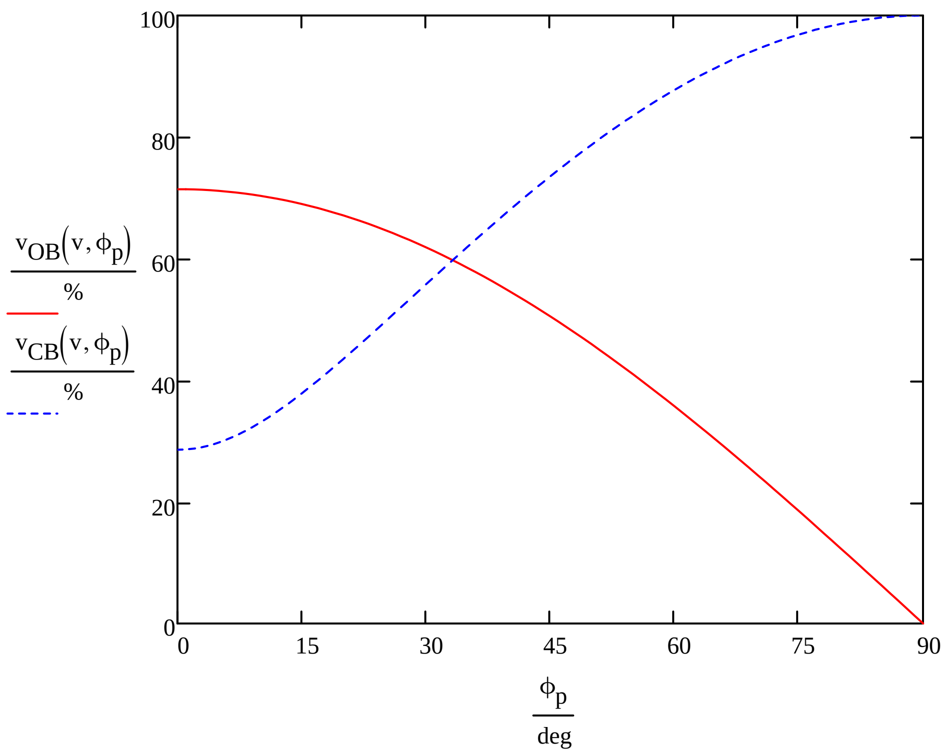
For example, for a 1/2-ball hit,

$$\phi\left(\frac{1}{2}\right) = 30 \cdot deg \qquad f(30 \cdot deg) = 50 \cdot \%$$

Here are the final rolling cue ball and object ball speeds vs. cut angle and ball-hit fraction, as compared to the orginal rolling cue ball speed:

$$v := 100\% \qquad \phi_p := 0\cdot deg, 1\cdot deg .. 90\cdot deg$$





For a 1/2-ball hit:

$$\phi := 30 \cdot deg$$

$$v_{OB}(v,\phi) = 61.859 \cdot \% \qquad v_{CB}(v,\phi) = 55.787 \cdot \%$$

The cut angle and ball-hit fraction resulting in equal final ball speeds, under ideal conditions are:

Given

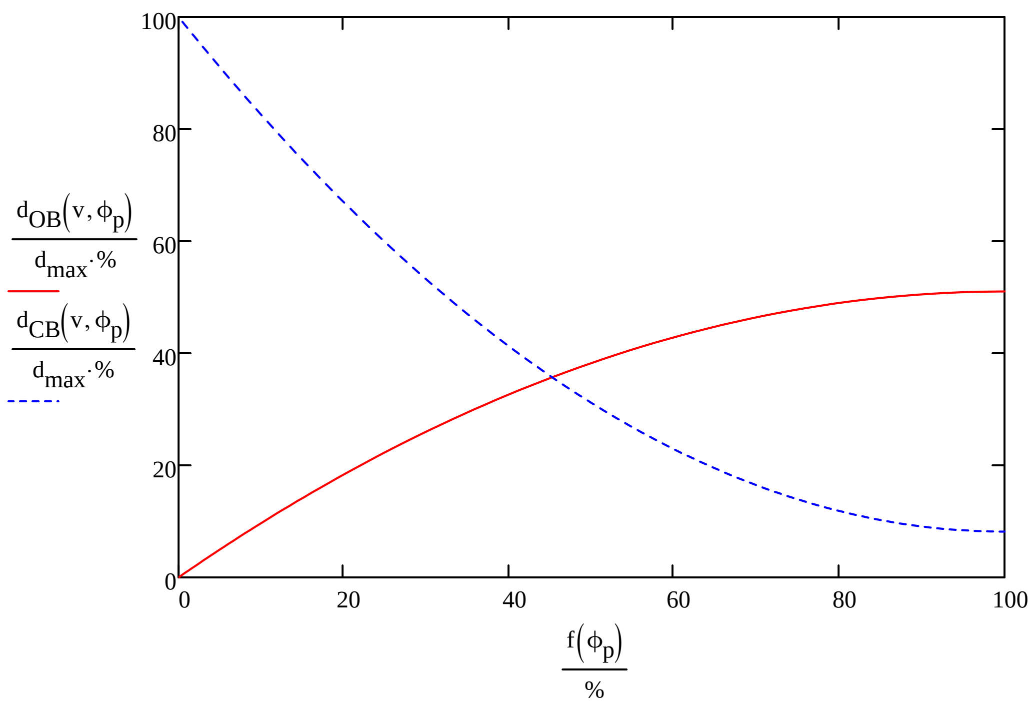
$$v_{OB}(v,\phi) = v_{CB}(v,\phi)$$

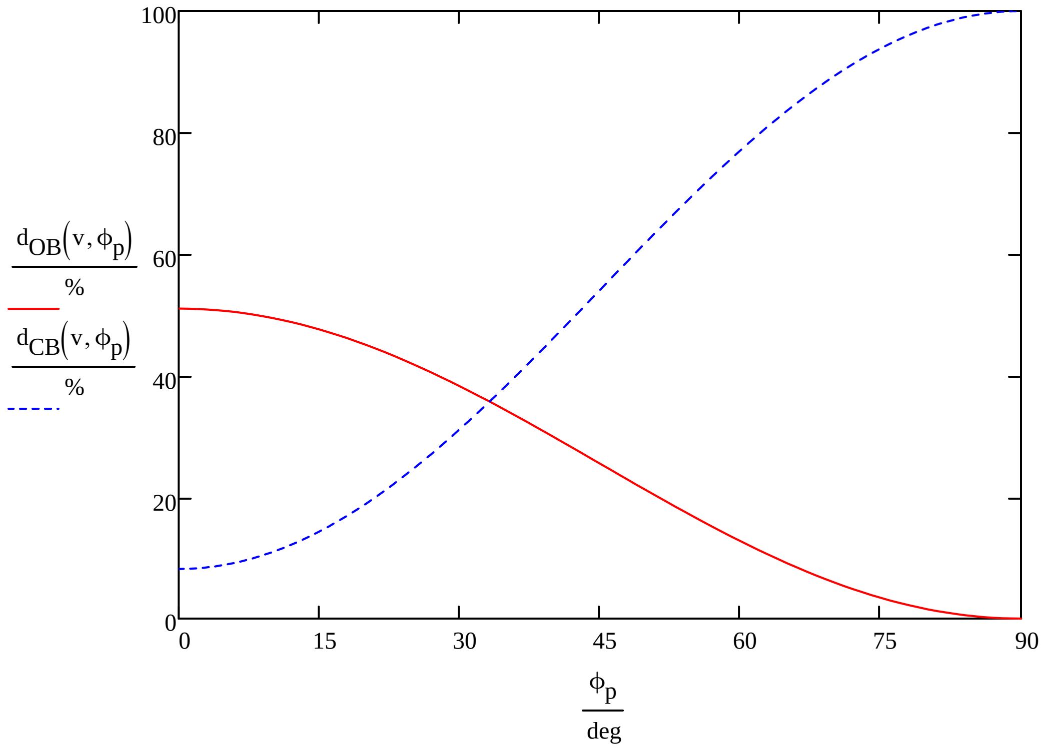
$$\phi_{equal} := \mathrm{Find}(\phi)$$

$$\phi_{equal} = 33.211 \cdot deg \qquad f\left(\phi_{equal}\right) = 45.228 \cdot \%$$

With constant rolling resistance, a rolling ball's travel distance is proportional to the square of the speed (as in TP 4.1); so the approximate relative distances traveled by each ball, compared to the distance the rolling cue ball would travel without collision (dmax), is:

$$v := 100\% \qquad d_{OB}(v,\varphi) := \left(v_{OB}(v,\varphi)\right)^2 \qquad d_{CB}(v,\varphi) := \left(v_{CB}(v,\varphi)\right)^2 \qquad d_{max} := d_{CB}(v, 90 \cdot deg)$$





Ideally, the ball speeds and distances traveled are the same at a little less than a 1/2-ball hit, which is slightly more than a 30-degree cut angle.

For a full-ball hit (100% hit or 0-degree cut angle), the post-impact, final rolling-ball speeds, as compared to the initial cue ball rolling speed are:

$$v_{CB}(v, 0) = 28.571 \cdot \% \qquad \frac{2}{7} = 28.571 \cdot \% \qquad\qquad v_{OB}(v, 0) = 71.429 \cdot \% \qquad \frac{5}{7} = 71.429 \cdot \%$$

It is clear from the equations at the top of this document where these proportions come from.

The ball travel distances, as compared to how far the initial rolling cue ball would travel are:

$$\frac{d_{CB}(v, 0)}{d_{max}} = 8.163 \cdot \% \qquad \left(\frac{2}{7}\right)^2 = 8.163 \cdot \% \qquad \frac{d_{OB}(v, 0)}{d_{max}} = 51.02 \cdot \% \qquad \left(\frac{5}{7}\right)^2 = 51.02 \cdot \% \qquad \frac{d_{OB}(v, 0)}{d_{CB}(v, 0)} = 6.25$$

So if a rolling cue ball hits an object ball squarely, the object ball will ideally travel a little more than 6 times farther than the cue ball after impact.

See the end of this document for graphs and distance proportions with non-ideal inelasticity and friction taken into consideration.

Here are travel distance ratios for various ball-hit fractions and cut angles (assuming perfect balls):

$f(\phi) := 1 - \sin(\phi) \qquad \phi(f) := \operatorname{asin}(1 - f)$

			approximate OB:CB distance ratio:
full-ball hit:			
$f_x := 1$	$\varphi_x := \phi(f_x) = 0 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 6.25$	6.5 : 1
3/4-ball hit:			
$f_x := \frac{3}{4}$	$\varphi_x := \phi(f_x) = 14.5 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 3.44$	7 : 2
1/2-ball hit (30-degree cut):			
$f_x := \frac{1}{2}$	$\varphi_x := \phi(f_x) = 30 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 1.23$	2.5 : 2
33.5-degree cut:			
$f_x := f(33.5 \cdot deg) = 44.806 \cdot \%$	$\varphi_x := 33.5 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 1$	1 : 1
1/4-ball hit:			
$f_x := \frac{1}{4}$	$\varphi_x := \phi(f_x) = 48.6 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 0.37$	1 : 2
3/16-ball hit (between 1/4 and 1/8):			
$f_x := \frac{3}{16}$	$\varphi_x := \phi(f_x) = 54.3 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 0.25$	1 : 4
1/8-ball hit:			
$f_x := \frac{1}{8}$	$\varphi_x := \phi(f_x) = 61 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 0.15$	1 : 6.5

If ball inelasticity and friction are taken into account (see TP A.5, TP A.6, and TP 4.1), the final speeds and directions can be found with:

$e := 0.94$ coefficient of restitution between balls

$\mu_{balls} := 0.06$ average coefficient of friction between the balls

final object ball velocity and speed

$$v_{OBn}(v,\phi) := \frac{5}{7}\cdot\frac{(1+e)}{2}\cdot\cos(\phi)\cdot v$$

$$v_{OBt}(v,\phi) := \frac{5}{7}\cdot\frac{(1+e)}{2}\cdot\mu_{balls}\cdot\cos(\phi)\cdot v$$

$$v_{OB}(v,\phi) := \sqrt{v_{OBn}(v,\phi)^2 + v_{OBt}(v,\phi)^2}$$

initial post-impact cue ball kinematics

$$\omega_{x0}(v,\phi) := \frac{v}{R}\cdot\left[\frac{5}{4}\cdot\mu_{balls}\cdot(1+e)\cdot\cos(\phi)^3 - 1\right]$$

$$\omega_{y0}(v,\phi) := \frac{v}{R}\cdot\left[\left[\frac{5}{4}\cdot\mu_{balls}\cdot(1+e)\cdot\sin(\phi)\cdot\cos(\phi)^2\right]\right]$$

$$v_{x0}(v,\phi) := \frac{v}{2}\cdot\sin(\phi)\cdot\cos(\phi)\cdot\left[1 + e - \mu_{balls}\cdot(1+e)\cdot\cos(\phi)\right]$$

$$v_{y0}(v,\phi) := \frac{v}{2}\cdot\left[\sin(\phi)^2\cdot\left[2 - \mu_{balls}\cdot(1+e)\cdot\cos(\phi)\right] + (1-e)\cdot\cos(\phi)^2\right]$$

$$v_{CBo}(v,\phi) := \sqrt{v_{x0}(v,\phi)^2 + v_{y0}(v,\phi)^2}$$ post-impact cue ball speed

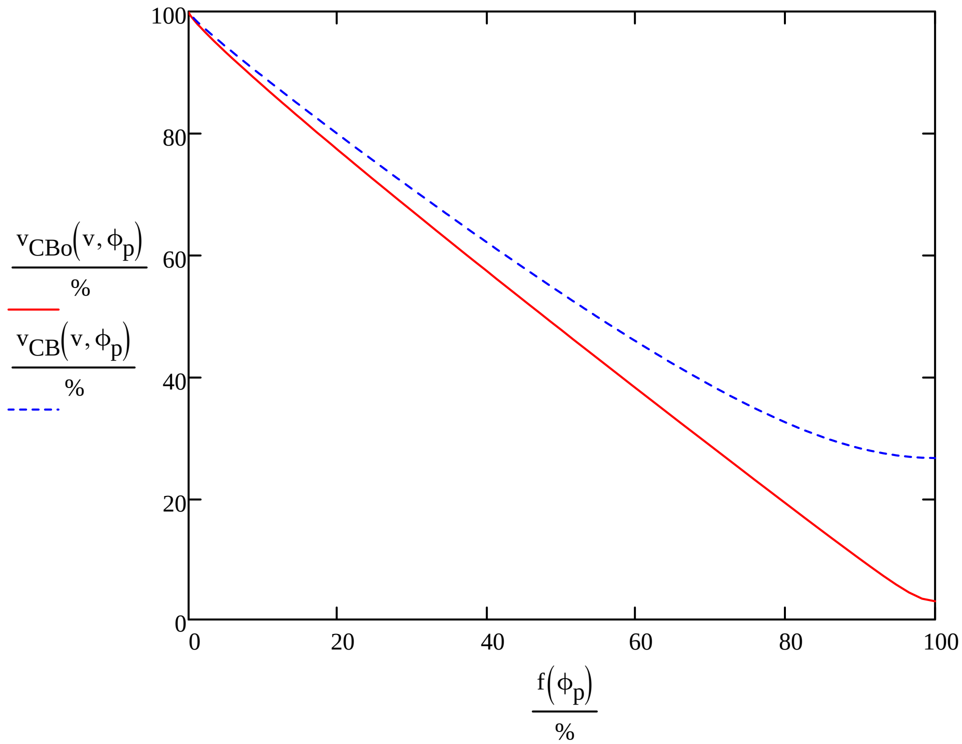
final cue ball velocity

$$v_{CBx}(v,\phi) := \frac{1}{7}\cdot\left(5\,v_{x0}(v,\phi) + 2\cdot R\cdot\omega_{y0}(v,\phi)\right)$$

$$v_{CBy}(v,\phi) := \frac{1}{7}\cdot\left(5\cdot v_{y0}(v,\phi) - 2\cdot R\cdot\omega_{x0}(v,\phi)\right)$$

$$v_{CB}(v,\phi) := \sqrt{v_{CBx}(v,\phi)^2 + v_{CBy}(v,\phi)^2}$$ final cue ball speed

This shows how the final CB speed compares with the post-impact speed for various ball-hit fractions:



So the CB picks up a little speed after impact for all ball-hit-fractions, except for a 0% ball-hit (90 degree cut angle).

To find the final ball angles, we first need to express the x and y components of the post-impact OB velocity (using the diagram in TP A.8):

$$v_{OBx}(v,\phi) := -v_{OBn}(v,\phi)\cdot\sin(\phi) + v_{OBt}(v,\phi)\cdot\cos(\phi)$$

$$v_{OBy}(v,\phi) := v_{OBn}(v,\phi)\cdot\cos(\phi) + v_{OBt}(v,\phi)\cdot\sin(\phi)$$

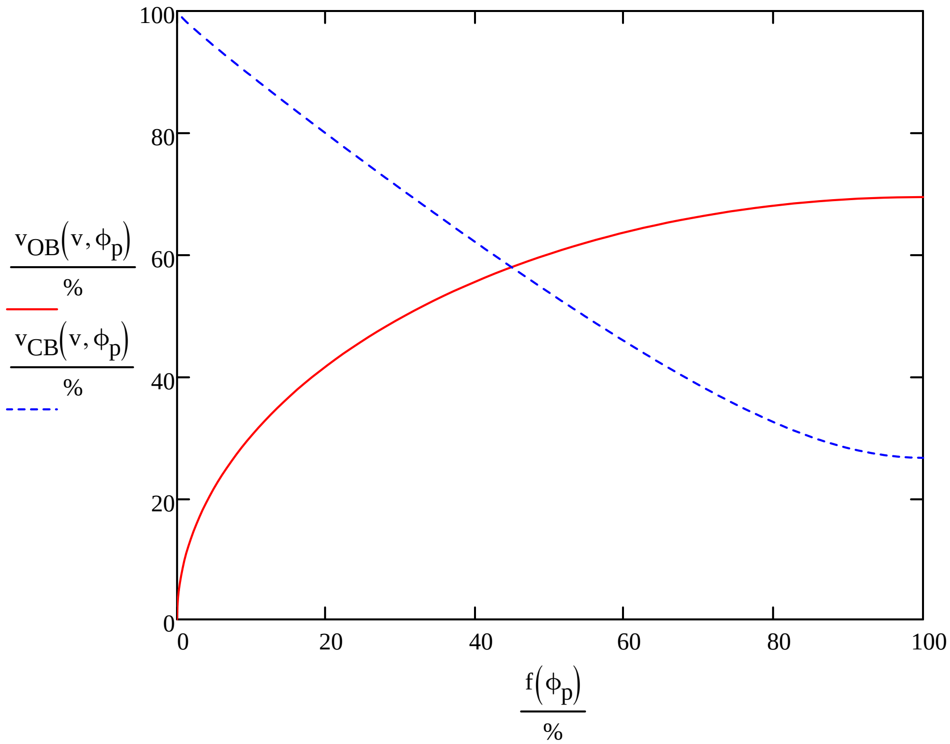
The final object ball angle (relative to the y-axis) is:

$$\theta_{OB}(v,\phi) := \operatorname{atan}\left(\frac{-v_{OBx}(v,\phi)}{v_{OBy}(v,\phi)}\right)$$

and the final cue ball angle (relative to the y-axis) is:

$$\theta_{CB}(v,\phi) := \operatorname{atan}\left(\frac{v_{CBx}(v,\phi)}{v_{CBy}(v,\phi)}\right)$$

This shows how the final OB and CB speeds vary with different ball-hit fractions:



For a 1/2-ball hit:

$$\phi := 30 \cdot deg$$

$$v_{OB}(v, \phi) = 60.111 \cdot \% \qquad v_{CB}(v, \phi) = 53.615 \cdot \%$$

The cut angle and ball-hit fraction resulting in equal final ball speeds are:

Given

$$v_{OB}(v, \phi) = v_{CB}(v, \phi)$$

$$\phi := \mathrm{Find}(\phi)$$

$$\phi = 33.477 \cdot deg \qquad f(\phi) = 44.839 \cdot \%$$

$$v_{OB}(v, \phi) = 57.895 \cdot \% \quad v_{CB}(v, \phi) = 57.895 \cdot \%$$

At this angle, the final OB and CB directions are:

$$\theta_{OB}(v, \phi) = 30.044 \cdot deg \qquad \theta_{CB}(v, \phi) = 33.409 \cdot deg$$

So, for about a 1/2-ball hit, the final CB and OB speeds are close to equal (about 58% of the original CB speed), and both balls leave at about the same angle (about 30 degrees).

The cut angle and ball-hit fraction resulting in equal final OB and CB angles are:

Given

$$\theta_{OB}(v,\phi) = \theta_{CB}(v,\phi)$$

$\phi := \text{Find}(\phi)$

$\phi = 36.114 \cdot \text{deg}$ $\quad f(\phi) = 41.06 \cdot \%$

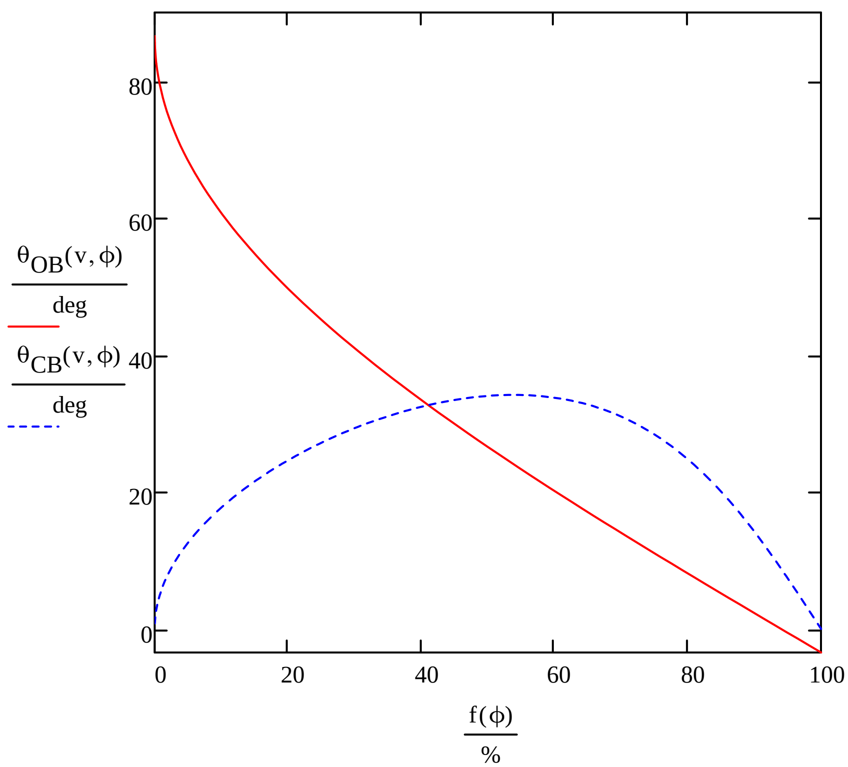
$\theta_{OB}(v,\phi) = 32.681 \cdot \text{deg}$ $\quad \theta_{CB}(v,\phi) = 32.681 \cdot \text{deg}$

At this angle, the final ball speeds are:

$v_{OB}(v,\phi) = 56.073 \cdot \%$ $\quad v_{CB}(v,\phi) = 61.097 \cdot \%$

So for about a 41% ball-hit fraction (36 degree cut angle), the balls leave at the same angle, and the final CB speed is a little faster than the OB speed (see graph below).

$\phi := 0 \cdot \text{deg}, 1 \cdot \text{deg} .. 90 \cdot \text{deg}$



To have the final CB and OB speeds and angles match as close as possible, we can minimize the following function (sum of squares of scaled differences):

$$\mathrm{ERROR}(\phi) := \left(\frac{v_{OB}(v,\phi) - v_{CB}(v,\phi)}{v_{CB}(v,\phi)}\right)^2 + \left(\frac{\theta_{OB}(v,\phi) - \theta_{CB}(v,\phi)}{\theta_{CB}(v,\phi)}\right)^2$$

$\phi := 30 \cdot \mathrm{deg}$ initial guess

Given

$$0 \cdot \mathrm{deg} < \phi < 90 \cdot \mathrm{deg}$$

$\phi := \mathrm{Minimize}(\mathrm{ERROR}, \phi)$

$\phi = 35.081 \cdot \mathrm{deg}$ $\quad f(\phi) = 42.527 \cdot \%$

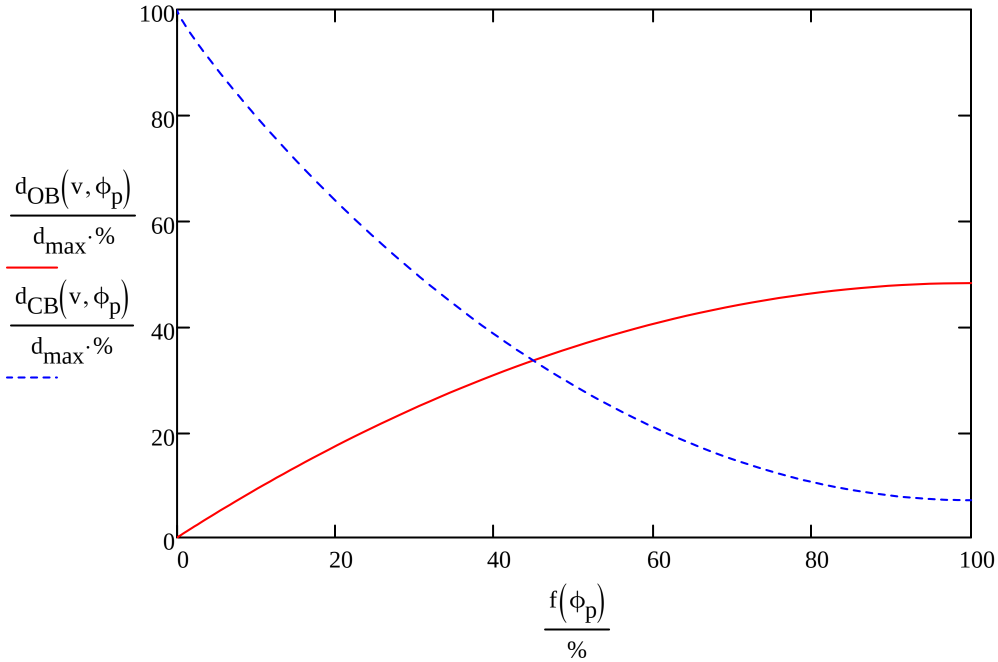
$$v_{OB}(v,\phi) = 56.801 \cdot \% \quad v_{CB}(v,\phi) = 59.848 \cdot \%$$

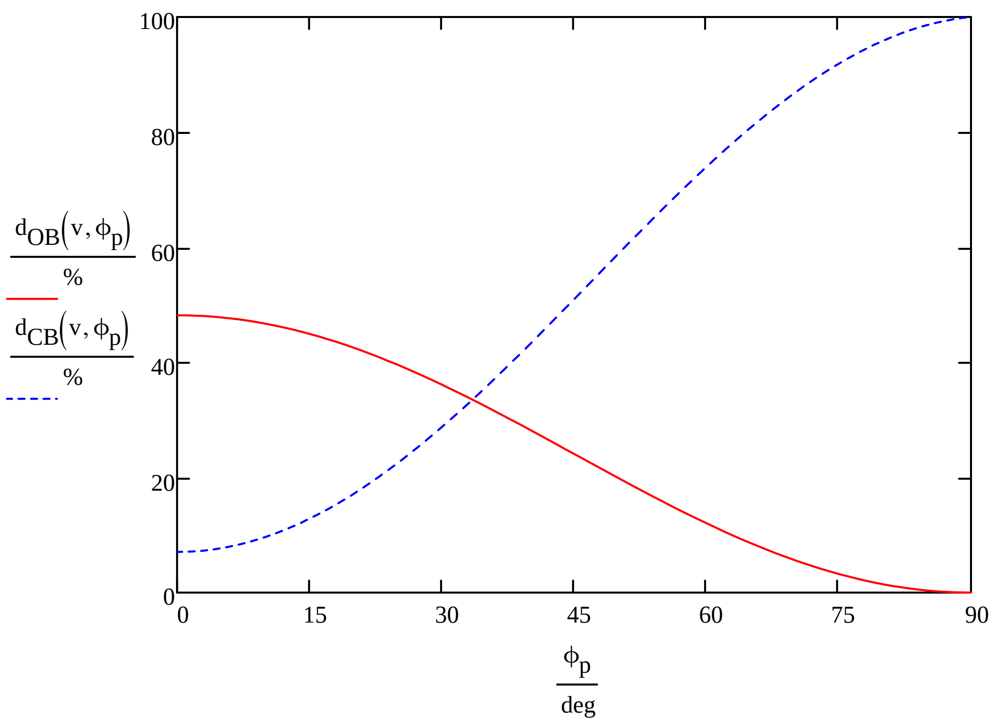
$$\theta_{OB}(v,\phi) = 31.647 \cdot \mathrm{deg} \quad \theta_{CB}(v,\phi) = 32.99 \cdot \mathrm{deg}$$

So a ball-hit fraction of about 43% (35 degree cut angle) results in the closest match of both CB and OB final speeds and angles.

With ball inelasticity and friction taken into account, here's how the ball travel distances vary with ball-hit fraction (f) and cut angle (φ), compared to the distance the rolling cue ball would travel without collision (dmax):

$v := 100\%$ $\qquad d_{OB}(v,\varphi) := \left(v_{OB}(v,\varphi)\right)^2$ $\qquad d_{CB}(v,\varphi) := \left(v_{CB}(v,\varphi)\right)^2$ $\qquad d_{max} := d_{CB}(v, 90\cdot deg)$





Find the cut angle and ball-hit fraction where the CB and OB have equal travel distances:

$\varphi := 35 \cdot deg$ initial guess

Given

$$d_{OB}(v, \varphi) = d_{CB}(v, \varphi)$$

$\varphi_{equal} := Find(\varphi) = 33.477 \cdot deg$ $\quad f(\varphi_{equal}) = 44.839 \cdot \%$

Here are travel distance ratios for various ball-hit fractions and cut angles (for typical imperfect balls):

$$f(\phi) := 1 - \sin(\phi) \qquad \phi(f) := asin(1 - f)$$

			approximate OB:CB distance ratio:
full-ball hit:			
$f_x := 1$	$\varphi_x := \phi(f_x) = 0 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 6.83$	7 : 1
3/4-ball hit:			
$f_x := \frac{3}{4}$	$\varphi_x := \phi(f_x) = 14.5 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 3.63$	7 : 2
1/2-ball hit (30-degree cut):			
$f_x := \frac{1}{2}$	$\varphi_x := \phi(f_x) = 30 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 1.26$	4 : 3
33.5-degree cut:			
$f_x := f(33.5 \cdot deg) = 44.806 \cdot \%$	$\varphi_x := 33.5 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 1$	1 : 1
1/4-ball hit:			
$f_x := \frac{1}{4}$	$\varphi_x := \phi(f_x) = 48.6 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 0.37$	1 : 2
3/16-ball hit (between 1/4 and 1/8):			
$f_x := \frac{3}{16}$	$\varphi_x := \phi(f_x) = 54.3 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 0.25$	1 : 5
1/8-ball hit:			
$f_x := \frac{1}{8}$	$\varphi_x := \phi(f_x) = 61 \cdot deg$	$\frac{d_{OB}(v, \varphi_x)}{d_{CB}(v, \varphi_x)} = 0.15$	1 : 10