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Frontal impact of rolling spheres

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A model of the inelastic collision between two spheres rolling along a horizontal track is presented, taking into account the effects of frictional forces at impact. The simple experiment described makes possible direct estimates of the coefficients of restitution and friction.

Experiments on collions are usually employed as examples of the laws of conservation; in particular, frontal and oblique collisions between rolling spheres have been frequently referred to in physics texts. The simple approach to the problem of colliding spheres assumes that no rolling occurs and omits the effects due to the frictional forces at impact. In this paper, we attempt to develop a more general approach to the problem of the frontal collisions between rolling spheres. It takes into account the inelasticity of impact and the frictional forces acting throughout the collision. These are expressed, respectively, in terms of the coefficient of restitution, e, and the coefficient of sliding friction, μ, in accordance with the general formulations of the two-body collision problem (Zukas 1982, Kane and Levinson 1985, Brach 1984). A simple experiment to verify the theoretical predictions of the model is described. This experiment leads to direct estimates of the coefficients of restitution and friction and illustrates the transition of rolling spheres from pure rolling to a combination of rolling and sliding.

Theory

Impact of rolling spheres. Let us consider two spheres, labelled 1 and 2, of equal radius, R_b, that collide while rolling along a horizontal rectangular track, with initial velocities u_j ($j=1,2$). At impact, each sphere is assumed to exert an impulsive force on the other at the point of contact, which can be expressed in terms of normal and tangential impulsive forces, as depicted in figure 1. The tangential impulsive forces will be considered as frictional forces acting during the time interval of impact.

If we denote by v_j ($j=1,2$) the velocities of the centres of mass of the spheres immediately after impact, then conservation of momentum can be expressed by

$$m_1u_1+m_2u_2=m_1v_1+m_2v_2 \qquad (1)$$

where m_j ($j=1,2$) are the masses of the spheres. On the other hand, the inelasticity of the impact is expressed in terms of the coefficient of restitution, e, by the equation

$$v_2-v_1=e(u_1-u_2) \qquad (2)$$

in which we assume, explicitly, that e is a constant that depends only on the materials of the spheres. This is a simplification valid under certain conditions (Goldsmith 1960).

Introducing the ratio of the masses, M, defined as $M=m_1/m_2$, we obtain from equations (1) and (2) for the velocities just after the impact:

$$v_1=\frac{M-e}{1+M}u_1+\frac{1+e}{1+M}u_2 \qquad (3)$$

$$v_2=M\frac{1+e}{1+M}u_1+\frac{1-eM}{1+M}u_2. \qquad (4)$$

The frictional forces must cause a change in the angular velocities of the spheres, from their initial values, u_j/R_e, to the values ω_j after the impact, which can be deduced from the integration of the equation $I_j\,d\omega_j=\pm\mu R_b m_j\,dv_j$, in which $I_j=2m_jR_b^2/5$ represents the moment of inertia of each sphere ($j=1,2$) and μ the coefficient of sliding friction between them, which is also regarded as a constant characteristic of the materials, and independent of velocity. R_e is the effective radius of each sphere, defined as the distance from its centre to the plane of contact with the track (see figure 2), that can easily be calculated from the radius, R_b, and the groove width, a, as $R_e=[R_b^2-(a/2)^2]^{1/2}$.

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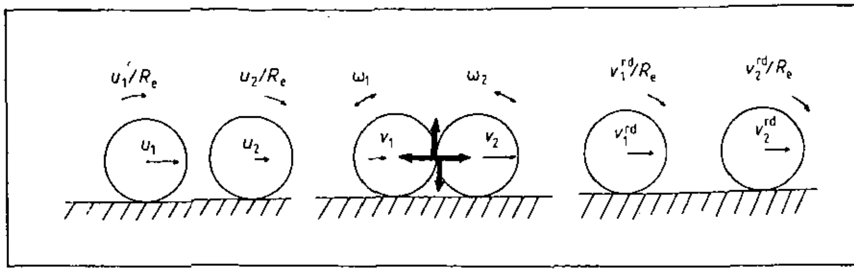

Figure 1. Schematic diagrams illustrating the frontal impact of rolling spheres.

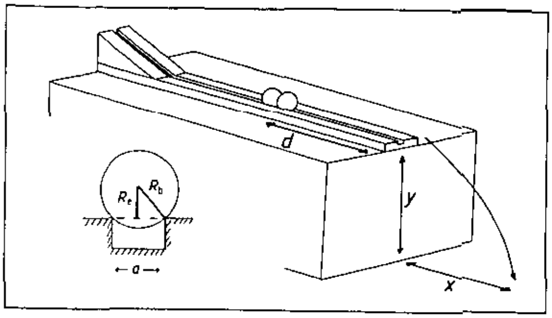

Figure 2. Experimental arrangement and cross-sectional view of the sphere on the track.

Thus, we can obtain

$$\omega_1 R_e = u_1 - \frac{5}{2}(R_e/R_b)\mu\,\frac{1+e}{1+M}(u_1-u_2) \tag{5}$$

$$\omega_2 R_e = u_2 - \frac{5}{2}(R_e/R_b)\mu M\,\frac{1+e}{1+M}(u_1-u_2) \tag{6}$$

Since $\omega_j \neq v_j/R_e$ it is obvious that, in general, the motion of the spheres after impact will be a combination of rolling and sliding.

From the above equations, the case in which one of the spheres is initially at rest can be described by making u_1 (or u_2) equal to zero; it is interesting to note that for an elastic collision ($e=1$) between spheres of equal masses ($M=1$), equations (1) and (2) predict, respectively, $v_2=0$ and $v_2=u_1$, i.e., the balls interchange their velocities, as is frequently cited in general textbooks. The impact of a rolling sphere against a vertical rigid surface is described by putting the m_1/m_2 ratio to zero; as expected, the model predicts that $v_1=-u_1$ for an elastic collision with a rigid barrier.

Motion of the spheres after the impact. After the impact, the frictional forces acting between the balls and the track cause changes in the linear and angular velocities given respectively by $m_j\,\mathrm{d}v_j/\mathrm{d}t = \pm\mu_j^s m_j g(R_b/R_e)$ and $I_j\,\mathrm{d}\omega_j/\mathrm{d}t = \pm\mu_j^s R_e m_j g(R_b/R_e)$, where μ_j^s denotes the coefficient of friction of the j-sphere with the track. The sign in these expressions depends on the direction of the frictional force, given, in turn, by the sign of $v_j - \omega_j R_e$.

By integration of these equations we can obtain the velocities of the spheres at a time t, corresponding to a combination of rolling and sliding motions. The pure rolling motion is reached when $v_j = \omega_j R_e$; therefore, the values of the velocities, v_j^{rd}, will be (introducing the ratio $R = R_e/R_b$)

$$v_1^{rd} = \left[\left(1 - \frac{5}{2}\mu R\frac{1+e}{1+M} + \frac{5}{2}R^2\frac{M-e}{1+M}\right)u_1 + \frac{5}{2}R\frac{1+e}{1+M}(R+\mu)u_2\right](1+5/2R^2)^{-1} \tag{7}$$

$$v_2^{rd} = \left[\frac{5}{2}RM\frac{1+e}{1+M}(R-\mu)u_1 + \left(1+\frac{5}{2}\mu RM\frac{1+e}{1+M} + \frac{5}{2}R^2\frac{1-eM}{1+M}\right)u_2\right] \times (1+5/2R^2)^{-1} \tag{8}$$

at times after the impact $t_2 = t^*(1-5\mu R/2)/\mu_1^s$ and $t_2 = t^*M(1+5\mu R/2)/\mu_2^s$, where

$$t^* = \frac{R(1+e/1+M)(u_1-u_2)}{g(1+5/2R^2)}. \tag{9}$$

Notice that the coefficients of friction between the spheres and the track are absent from equations (5) and (6), which can be obtained directly by integration of $I_j\,\mathrm{d}\omega_j = -R_e m_j\,\mathrm{d}v_j$. It should be noted that in this treatment we have neglected the rolling friction terms required in more detailed approaches to the problem (Doménech *et al* 1987).

Experimental procedure

Verification of the model has the intrinsic difficulty associated with the experimental determination of the velocities, requiring relatively complicated equipment. To overcome this difficulty we can study the case in which one sphere collides with the second sphere (initially) at rest. The balls collide on a rectangular track placed on a laboratory bench, with the lower end of the track flush with the edge of the bench. After moving off the track the balls fell through the air, finally striking the floor at a horizontal distance x and a vertical distance y from the end of the track.

The points of impact of the balls with the floor can be determined using carbon paper. Neglecting air resistance, the trajectory is a parabola of form $y=gx^2/2v^2$, and so the velocity of the sphere at the end of the track can be calculated from the distances x and y.

For simplicity, we can obtain the ratios v_1/u_1 and v_2/u_1 directly from measurements of the horizontal distances covered by ball 1 without impact (x_0) and by the balls 1 and 2 after the impact (x_1,x_2). Obviously, the velocity of the incident sphere at impact, u_1, must be constant; this requirement is accomplished by using a slanted auxiliary track placed over the horizontal grooved track.

When the experiment was carried out, the students found a relatively surprising result: the ratios of the velocities depended on the location of the point of impact of the spheres with respect to the end of the track (see figure 2). This fact can be explained in terms of the transition from the initial rolling and sliding motion of the spheres to the pure rolling motion due to friction with the track. Thus, the ratios of the velocities must vary from

$$v_1/u_1=(M-e)/(1+M) \tag{10}$$

$$v_2/u_1=M(1+e)/(1+M) \tag{11}$$

immediately after the impact to

$$v_1^{\text{rd}}/u_1=\left(1-\frac{5}{2}\mu R\frac{1+e}{1+M}+\frac{5}{2}R^2\frac{M-e}{1+M}\right)\times(1+5/2R^2)^{-1} \tag{12}$$

$$v_2^{\text{rd}}/u_1=\left(\frac{5}{2}RM\frac{1+e}{1+M}(R-\mu)\right)(1+5/2R^2)^{-1} \tag{13}$$

corresponding to the pure rolling motion. Equations (10) and (11) imply that the distances, d_j, of the spheres from the end of the track are zero. When the distance d_j is larger than a given limiting value, d_j^{L}, pure rolling motion obtains, and the velocities of the spheres will be described by equations (12) and (13). For intermediate distances ($0<d_j<d_j^{\text{L}}$) the square of the velocities must be linearly dependent on the distance d_j, i.e.

$$v_j^2(d_j)=v_j^2(d_j=0)\pm 2\mu_j^{\text{s}}gd_j/R. \tag{14}$$

Once pure rolling motion has been reached ($d_j>d_j^{\text{L}}$) the velocity of the spheres along the track remains practically constant. Deceleration due to rolling friction and air resistance are negligible given our experimental conditions.

Results and discussion

Aluminium, brass and steel spheres of 2.50 cm diameter were employed in collisions on an aluminium track with groove width 1.00 cm. The velocity of the incident ball was adjusted to 0.75 ± 0.10 m s^{-1} by using a coupled slanted track. The end of the horizontal track was placed at a height of 75 cm above the floor. The data were obtained by varying the distance d and measuring the horizontal distances x_0, x_1, x_2, covered by the spheres. Experiments were repeated five times in order to obtain the x_j distances; in all cases, the dispersion of the x_j values was less than 2%.

The experimental results agreed satisfactorily with the proposed model under our experimental conditions. In figure 3 we have plotted the values of x_j/x_0 in terms of the values of d_j for the collisions between steel (s), brass (b) and aluminium (a) spheres, in all cases with $M=1$. As can be observed, the ratio x_1/x_0 remains practically constant while the ratio x_2/x_0 decreases when the distance d_2 increases, until an abrupt change occurs for a distance d_2^{L}, when pure rolling motion commences.

Figure 3. Plots of the experimental values of x_j/x_0 versus the distances d_j for the collisions of different pairs of spheres.

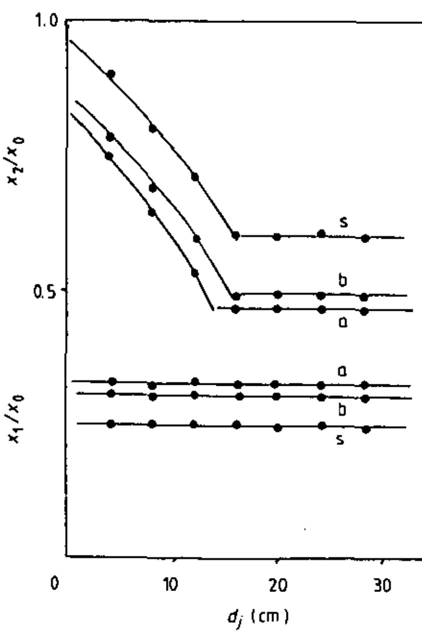

Table 1. Coefficients of restitution and friction determined for several pairs of spheres.

Materials (1–2)	*M*	*e*	μ	μ_2^s
aluminium–aluminium	1.00	0.69 ± 0.02	0.17 ± 0.04	0.17 ± 0.03
brass–brass	1.00	0.72 ± 0.02	0.16 ± 0.04	0.16 ± 0.03
steel–steel	1.00	0.95 ± 0.02	0.11 ± 0.03	0.15 ± 0.03
brass–steel	1.05	0.72 ± 0.02	0.14 ± 0.04	0.15 ± 0.03
brass–aluminium	3.16	0.61 ± 0.02	0.12 ± 0.03	0.15 ± 0.03
steel–aluminium	3.01	0.55 ± 0.02	0.14 ± 0.03	0.16 ± 0.03

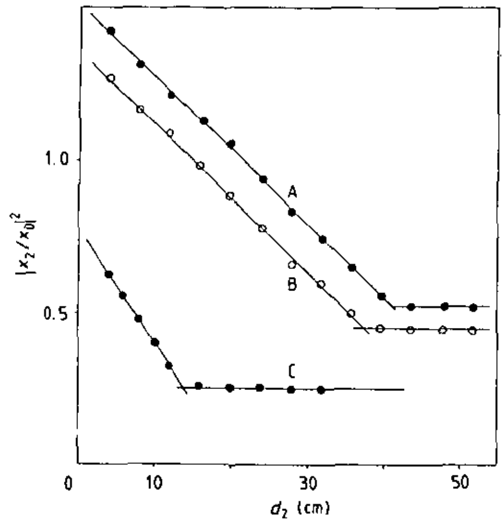

Figure 4. Graphical representation of $(x_2/x_0)^2$ versus d_2 for the collisions of brass–aluminium (A), steel–aluminium (B) and brass–steel (C) spheres.

As shown in figure 4, $(x_2/x_0)^2$ varies linearly with d_2, for small values of this distance, allowing e to be obtained from the intercept using equation (11) and μ_2^s from the slope using equation (14) in the corresponding least-squares adjustment.

For distances $d_2 > d_2^L$, the quotient x_2/x_0 remains practically constant, as can be seen also from figures 3 and 4. This limiting value of x_2/x_0 permits an estimate to be made of the coefficient of friction between the spheres, inserting into equation (13) the value of e previously calculated.

For ball 1, only the rolling motion is recorded, as shown in figure 3. The experimental values of x_1/x_0 are in excellent agreement with those calculated by substituting the above values of μ and e into equation (12). This fact provides further support for the validity of the model. The values of the coefficients of restitution and friction calculated for several pairs of materials are listed in table 1.

The students can compare the experimental results with successive theoretical formulations of the problem. Thus, in a first step, the solution of the equations of the frontal elastic collisions without frictional effects can be obtained. In a second approach, the inelasticity of the impact is included, yielding a satisfactory description of the velocities immediately after the impact. Finally, frictional effects between the spheres and in the sphere–track contact are incorporated into the model to account for the observed results both qualitatively and quantitatively.

Concluding remarks

The study of collisions exemplifies one of the most significant aspects of the scientific methodology: the introduction of successive approaches to a general model to obtain progressive improvements in the description of a particular problem.

The present formulation can be considered as a plausible description of the frontal impact of rolling spheres, and allows a simple mechanics experiment illustrating the classical laws of impact and friction, yielding rapid estimates of the coefficients of restitution and friction. In addition, the transition of a rolling sphere from pure rolling to a combination of rolling and sliding can also be studied.

References

Brach R M 1984 *J. Appl. Mech.* **51** 164
Doménech A, Doménech M T and Cebrián J 1987 *Am. J. Phys.* **55** 231
Goldsmith W 1960 *Impact* (London: Arnold)
Kane T R and Levinson D A 1985 *Dynamics: Theory and Applications* (New York: McGraw-Hill)
Zukas J A 1982 *Impact Dynamics* (New York: Wiley)